8-2011

Circle Packings on Affine Tori

Christopher Thomas Sass
csass@utk.edu

To the Graduate Council:

I am submitting herewith a dissertation written by Christopher Thomas Sass entitled "Circle Packings on Affine Tori." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Kenneth Stephenson, Major Professor

We have read this dissertation and recommend its acceptance:

Carl Sundberg, James Conant, Jesse Poore

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Circle Packings on Affine Tori

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Christopher Thomas Sass
August 2011

Dedication

To Bethann.

Acknowledgments

I would like to express my sincere gratitude to my advisor, Dr. Stephenson. I especially appreciate his mathematical curiosity, enthusiasm, and knowledge, as well as his generosity in sharing his time. He has offered ongoing encouragement and support in many forms over a period of several years.

It has been my good fortune to have studied under a number of exceptional professors during my time at the University of Tennessee. In particular, Dr. Stephenson, Dr. Wade, and Dr. Plaut were truly inspiring.

It is impossible to overestimate the importance of daily interactions with the friendly and intelligent graduate students in mathematics during my time at the University of Tennessee. I owe special thanks to Ben Lynch, Kirill Yakovlev, Justin Grieves, Jon Gray, John Engbers, and Jay Wilkins.

Pam Armentrout is an amazing source of knowledge concerning the math department and the university. She has helped me in countless ways during the past six years.

Regular breaks from the intense work of graduate school have been a necessity for me and have come in the form of outings to the Smoky mountains. I want to thank Bethann Bowman and Brad Crockett for hiking hundreds of miles with me during the past six years. I also want to thank the following members of the Smoky Mountains Hiking Club for some particularly adventurous off-trail hikes: Jenny Bennett, Hiram Rogers, Mark Shipley, Ed Fleming, Alan Householder, Greg Hoover, Greg Harrell, and David Smith.

Finally, I want to thank my wife, Bethann Bowman, for her unfailing love and support during the past six years.

The pictures of circle packings in this thesis were created using Kenneth Stephenson's `CirclePack` software.

Abstract

This thesis is a study of circle packings for arbitrary combinatorial tori in the geometric setting of affine tori. Certain new tools needed for this study, such as face labels instead of the usual vertex labels, are described. It is shown that to each combinatorial torus there corresponds a two real parameter family of affine packing labels. A construction of circle packings for combinatorial fundamental domains from affine packing labels is given. It is demonstrated that such circle packings have two affine side-pairing maps, and also that these side-pairing maps depend continuously on the two real parameters.

Contents

List of Figures

Chapter 1

Introduction

1.1 Theoretical Motivation

A circle packing is an arrangement of circles according to a specified pattern of external tangencies. Of course, a circle packing must be situated in a geometric space. A circle packing that "fills up" such a geometric space is a **maximal packing**. Maximal packings on the Riemann sphere were studied in Koebe (1936), Andreev (1970a), Andreev (1970b), and Thurston (1985). Maximal packings in the hyprerbolic disc were addressed by Thurston (1985). And results concerning maximal packings in the euclidean plane were established by Beardon and Stephenson (1990), Rodin (1991), and He and Schramm (1993).

The existence and uniqueness of maximal packings in the much more general geometric setting of Riemann surfaces (also known as "conformal surfaces") were shown in Beardon and Stephenson (1990). One feature of such packings is their rigidity: for instance, given a specified pattern of tangencies suitable for a torus, there exists exactly one conformal torus (among uncountably many such tori) on which a maximal packing for the given pattern can be realized.

A further extension of the geometric setting was explored in Mizushima (2000). In this work, circle packings on affine tori (these are generalizations of conformal tori) are examined. An affine torus is specified by an affine parameter c in the complex plane and a Teichmüller parameter ω in the upper half plane. The major results of this paper are:

1. The existence of circle packings on affine tori.

2. For a certain pattern of tangencies suitable for a torus, there exists a real two-dimensional family of affine tori on which a maximal packing for the given pattern can be realized.

3. For this two-dimensional family, the projection $(\omega, c) \mapsto \omega$ is surjective.

4. The projection $(\omega, c) \mapsto \omega$ is 2-to-1 except at a single point.

A consequence of the second result is that circle packings on affine tori are flexible, in contrast to the rigidity of circle packings on conformal tori. The limitation of the Mizushima paper is that only one pattern of tangencies is considered, and it is a degenerate pattern. The circle packings in Mizushima (2000) turn out to consist of a single circle with three points of self-tangency. In such a simple setting it is possible to work with explicit formulas relating to the affine and Teichmüller parameters.

The theoretical motivation for the work in this thesis is the extension of the study of circle packings on affine tori to circle packings for **any** pattern of tangencies suitable for a torus. In order to deal with this more general and complex situation, it is necessary to develop and utilize techniques completely different from those used by Mizushima. These new techniques are employed to show that the first two of Mizushima's results hold for any pattern suitable for a torus. Whether the same is true for the third and fourth results are open questions. But theoretical results concerning continuity and experimental observations of certain monotonicities should be useful for further study of the open questions.

1.2 Practical Motivation

The papers Kojima et al. (2006) and Kojima et al. (2003) address circle packings in the context of projective structures on Riemann surfaces (these are also generalizations of Riemann surfaces). While the main concern of the authors is with surfaces of genus $g > 1$, they address circle packings on affine tori as a special case. It is shown that circle packings on projective Riemann surfaces and on affine tori may be parametrized by semi-algebraic sets in some $\mathbb{R}^n$. The authors study the local topology of the parameter space using techniques such as hyperbolic Dehn surgery.

The practical motivation for the work in this thesis is the study circle packings on affine tori from an elementary point of view, using techniques developed by experimental computations of circle packings on affine tori. From this point of view the necessary distinction between circle packings on affine tori and circle packings on projective surfaces of higher genus is naturally emphasized, and the special geometric features of circle packings on affine tori are readily apparent.

Chapter 2

Preliminaries

2.1 Projective and Affine Structures on Riemann Surfaces

Let M be a compact Riemann surface represented as the quotient of its universal cover $\widetilde{M}$ by the group of covering translations Γ. A **projective structure** on M is an analytic local homeomorphism $f : \widetilde{M} \to \mathbb{P}$ such that for each $A \in \Gamma$ there exists an automorphism of the Riemann sphere $\sigma_A : \mathbb{P} \to \mathbb{P}$ such that:

$$f \circ A = \sigma_A \circ f.$$

Projective structures f and g on M are considered equivalent if there is an automorphism ρ of $\mathbb{P}$ such that $f = \rho \circ g$.

An **affine structure** on M is an analytic local homeomorphism $f : \widetilde{M} \to \mathbb{C}$ such that for each $A \in \Gamma$ there exists an automorphism of the complex plane $\sigma_A : \mathbb{C} \to \mathbb{C}$ such that:

$$f \circ A = \sigma_A \circ f.$$

Affine structures f and g on M are considered equivalent if there is an automorphism ρ of $\mathbb{C}$ such that $f = \rho \circ g$.

It is a classical result that the only Riemann surfaces supporting affine structures are genus 1 surfaces (that is, tori): see Milnor (1958). Fix a conformal torus T. Its universal covering space $\widetilde{T}$ may be identified with $\mathbb{C}$. Its group of covering transformations is a free abelian group of rank 2. A marking on M is then simply a choice of two free generators $A, B \in \Gamma$. Such markings can always be given as $A(z) = z + 1$

and $B(z) = z + \omega$, where ω is a complex number in the upper half plane $\mathbb{H}$. It is known that in this manner the Teichmüller space of marked conformally equivalent tori is identified with $\mathbb{H}$: see Gunning (1981).

Fix a marked conformal torus $T(\omega)$. An affine structure f on M is then an analytic local homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$,

$$f(z+1) = a_A f(z) + b_A, \tag{2.1}$$

$$f(z+\omega) = a_B f(z) + b_B, \tag{2.2}$$

for some $a_A, b_A, a_B, b_B \in \mathbb{C}$ with $a_A, a_B \neq 0$. Since f is locally one-to-one, $f' \neq 0$ on $\mathbb{C}$. And since $\omega \in \mathbb{H}$ it follows from 2.1 and 2.2 that $f''/f' \equiv c$ on $\mathbb{C}$. By passing to equivalent affine structures, we may suppose that if $c = 0$ then $f(z) = z$ and if $c \neq 0$ then $f(z) = e^{cz}$. It is known that in this manner the space of affine structures on $T(\omega)$ is identified with $\mathbb{C}$: see Gunning (1981).

It follows that an affine torus may be specified by giving its **Teichmüller parameter** $\omega \in \mathbb{H}$ and its **affine parameter** $c \in \mathbb{C}$. It is the goal of this thesis to establish the existence and uniqueness of circle packings on affine tori and to explore certain properties of such circle packings.

2.2 Conditions on Combinatorics

The combinatorics of a circle packing are encoded as a simplicial 2-complex K. As discussed in Stephenson (2005), the following conditions on K are necessary and sufficient for K to represent a triangulation of an oriented surface.

1. K is connected.

2. Every edge belongs to either one face (boundary edge) or two faces (interior edge).

3. Every vertex of v belongs to at most finitely many faces, and these form an ordered chain in which each face shares an edge from v with the next.

4. Every vertex belongs either to no boundary edge (interior vertex) or to exactly two boundary edges (boundary vertex).

5. Any two faces are either disjoint, share a single vertex, or share a single edge.

6. An order may be assigned to the vertices in every face in such a way that any pair of faces intersecting in an edge will induce opposite orientations on that edge.

The combinatorics in Mizushima (2000) consist of one vertex, three edges, and two faces. All three edges are shared by the two faces, so the fifth condition fails for Mizushima's combinatorics.

Remark. Throughout this thesis it is assumed that all complexes satisfy these conditions.

2.3 Definitions and Notation

Let K be a combinatorial torus. The usual circle packing approach (as in Stephenson (2005)) is to let a label R for K be a tuple of positive real numbers. The size of the tuple is equal to the number of vertices of K. The tuple entries $R(v)$, where v is a vertex of K, are thought of as putative circle radii. If $f = \langle u, v, w\rangle$ is a face of K, and if R is taken to be a euclidean label, the angle $\alpha_R(u; v, w)$ at the vertex u in the face $\langle u, v, w\rangle$ relative to the label R is given by the euclidean law of cosines:

$$\alpha_R(u; v, w) = \arccos \frac{(R(u) + R(v))^2 + (R(u) + R(w))^2 - (R(v) + R(w))^2}{2(R(u) + R(v))(R(u) + R(w))}. \tag{2.3}$$

An important tool in the generalization of the geometric setting of circle packings from conformal to affine tori is the following generalization of the notion of a label.

Definition 2.3.1. A **face label** R for K is a tuple of positive real numbers, one for each pair (v, f) of vertices $v \in K$ and faces f containing the vertex v. The tuple entries will be denoted $R_f(v)$.

A euclidean face label is one in which the angle $\alpha_R(u, f)$ at the vertex u in the face $f = \langle u, v, w\rangle$ relative to the label R is given by the euclidean law of cosines as in (2.3), but with '$R_f(u)$', '$R_f(v)$', and '$R_f(w)$' repacing '$R(u)$', '$R(v)$', and '$R(w)$' respectively. All face labels considered in this thesis will be taken to be euclidean face labels.

Notation. If v is a vertex of K, let $F(v)$ be the subcomplex of faces of K containing v.

Using this notation, the angle sum $\theta_R(v)$ relative to a face label R at a vertex v is given by:

$$\theta_R(v) = \sum_{f \in F(v)} \alpha_R(v, f).$$

Definition 2.3.2. An **edge path** $\Gamma = \{v_1, ..., v_n\}$ in K is an ordered collection of vertices of K such that the $\langle v_i, v_{i+1}\rangle$ is an edge of K for $i = 1, ..., n-1$. An edge path is **simple** if $v_i = v_j \Rightarrow i = j$ or $i, j \in \{1, n\}$. An edge path is **closed** if $v_1 = v_n$.

Remark. By a slight abuse of terminology, we will refer to the edges joining successive vertices of an edge path Γ as the edges of Γ.

Definition 2.3.3. A face f is on the **left side** of the simple edge path $\Gamma = \{v_1, ..., v_n\}$ if $f = \langle v_i, v_{i+1}, w\rangle$ for some $i \in \{1, ..., n-1\}$ and for some $w \in K$. In other words, an edge of f is in Γ and the orientations of Γ and f are compatible. Similarly, a face f' is on the **right side** of Γ if an edge of f' is in Γ and the orientations of Γ and f' are not compatible.

Remark. Since K has no boundary, each edge of K belongs to exactly two faces of K. So each edge of an edge path belongs to exactly one face on the left side of the edge path and one face on the right side of the edge path.

Definition 2.3.4. If $\Gamma_1 = \{v_1, ..., v_n\}$ and $\Gamma_2 = \{w_1, ..., w_m\}$ are edge paths in K such that $v_n = w_1$, their **concatentation** is the edge path $\Gamma_1 * \Gamma_2 = \{v_1, ..., v_n, w_2, ..., w_m\}$.

Definition 2.3.5. An edge path Γ in K is a **fundamental pair** for K if

1. $\Gamma = \Gamma_1 * \Gamma_2$ where Γ_1 and Γ_2 are simple closed edge paths in K sharing a single vertex (the **corner** of Γ),
2. A combinatorial cut along Γ results in a combinatorial closed disc (a **fundamental domain** for K).

Definition 2.3.6. A face label R for K satisfies the **packing condition** if all angle sums relative to R are 2π.

Definition 2.3.7. Let v be a vertex of K, and let $f_1 = \langle v, w_1, w_2\rangle$, $f_2 = \langle v, w_2, w_3\rangle$, $\ldots$, $f_n = \langle v, w_n, w_{n+1}\rangle$ be the faces of $F(v)$ (where $w_{n+1} = w_1$). Let R be a face label for K. For $i = 1, \ldots, n$ let

$$l_i = S_{f_i}(v) + S_{f_i}(w_{i+1}),$$
$$r_i = S_{f_i}(v) + S_{f_i}(w_i).$$

The face label R for K is **weakly consistent at** v if

$$\frac{l_1 \cdot l_2 \cdots l_n}{r_1 \cdot r_2 \cdots r_n} = 1.$$

The face label R for K is **weakly consistent** if it is weakly consistent for all vertices of K.

Remark. If the face label R is weakly consistent, $a > 0$, and $f = \langle u, v, w \rangle$ is a face of K, then scaling the label R for face f by a (that is, replacing $R_f(u)$, $R_f(v)$, and $R_f(w)$ by $a \cdot R_f(u)$, $a \cdot R_f(v)$, and $a \cdot R_f(w)$ respectively) preserves weak consistency.

If the faces f_i are laid out as euclidean triangles, then l_i is the length of the edge $\langle v, w_{i+1} \rangle$, and r_i is the length of the edge $\langle v, w_i \rangle$. Weak consistency is equivalent to the following: if face f_1 is laid out, and then successive faces of $F(v)$ are scaled so that the edge shared with the previous face has the same length as in the previous face, then when the final face f_n is scaled its edge lengths agree with *both* of its neighboring faces in $F(v)$. Figure 2.1 shows an example of the result of this process when a face label fails to be weakly consistent at v: scaling the final face f_4 so that the edge shared with the previous face f_3 has the same length as in the previous face fails to result in agreement in edge lengths for the edge shared by the final face f_4 and the first face f_1.

Definition 2.3.8. Let $\langle u, v \rangle$ be an edge of K shared by faces f and g. A face label R for K is **strongly consistent at the edge** $\langle u, v \rangle$ if

$$\frac{R_f(u)}{R_f(v)} = \frac{R_g(u)}{R_g(v)}.$$

The face label R for K is **strongly consistent** if it is strongly consistent for all edges of K.

Remark. If the face label R is strongly consistent, $a > 0$, and $f = \langle u, v, w \rangle$ is a face of K, then scaling the label R for face f by a preserves strong consistency.

Lemma 2.3.9. *Let $\langle u, v \rangle$ be an edge of K shared by faces f and g. Supose that the face label R is strongly consistent at the edge $\langle u, v \rangle$. Then the following are equivalent:*

1. $aR_f(u) = R_g(u)$,

2. $aR_f(v) = R_g(v)$,

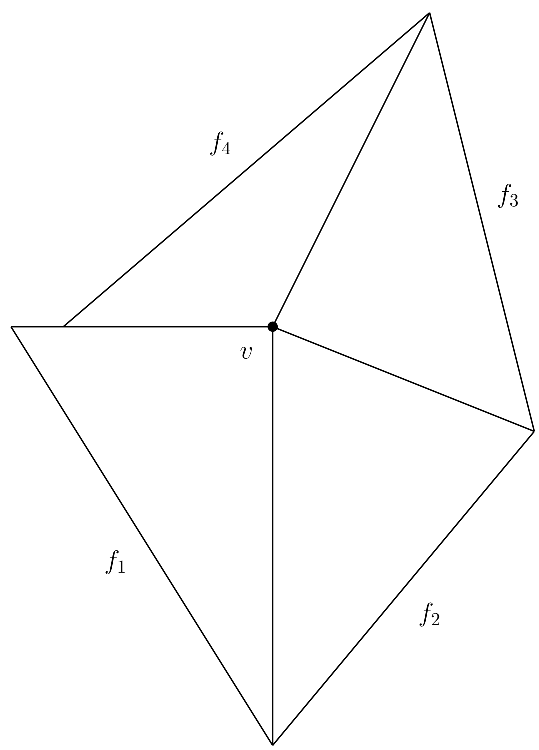

Figure 2.1: Failure of weak consistency at v

3. $a(R_f(u) + R_f(v)) = R_g(u) + R_g(v)$,

where $a > 0$.

Proof. (1) $\Leftrightarrow$ (2) is immediate from the definition of strong consistency. And (2) $\Rightarrow$ (3) is clear since (2) $\Rightarrow$ (1). To show (3) $\Rightarrow$ (2), suppose that (2) holds. Then by strong consistency and supposition:

$$\begin{aligned}
a(R_f(u) + R_f(v)) &= R_g(u) + R_g(v), \\
aR_f(v)(R_f(u) + R_f(v)) &= R_f(v)R_g(u) + R_f(v)R_g(v), \\
aR_f(v)(R_f(u) + R_f(v)) &= R_f(u)R_g(v) + R_f(v)R_g(v), \\
aR_f(v)(R_f(u) + R_f(v)) &= (R_f(u) + R_f(v))R_g(v), \\
aR_f(v) &= R_g(v).
\end{aligned}$$ $\square$

So strong consistency implies that if faces f and g share an edge and are laid out as euclidean triangles, then the same scaling factor $a > 0$ for f will give a multiplied by the f-length of $\langle u, v\rangle$ equals the g-length of $\langle u, v\rangle$ **and** $a \cdot R_f(u) = R_g(u)$. Figure 2.2 shows an example of failure of weak consistency. If $f = \langle u, v, w_1\rangle$ and $g = \langle v, u, w_2\rangle$, then in Figure 2.2 the shared edge has equal f-length and g-length $(R_f(u) + R_f(v) = R_g(u) + R_g(v))$, but $R_f(u) \neq R_g(u)$.

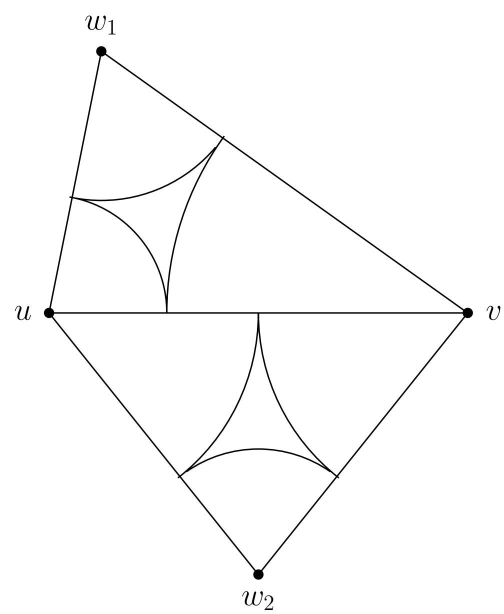

Figure 2.2: Failure of strong consistency at $\langle u, v\rangle$

Lemma 2.3.10. *If a face label R for K is strongly consistent, then it is also weakly consistent.*

Proof. Fix a vertex v in K, and use the notation of Definition 2.3.7 for the faces of $F(v)$. Scale face f_2 so that $R_{f_2}(v) + R_{f_2}(w_2) = R_{f_1}(v) + R_{f_1}(w_2)$. By Lemma 2.3.9 it follows that $R_{f_2}(v) = R_{f_1}(v)$. Proceed around the faces of $F(v)$, scaling successive faces so that the edge shared with the previous face has the same length as in the previous face. By Lemma 2.3.9 it follows that $R_{f_n}(v) = \cdots = R_{f_2}(v) = R_{f_1}(v)$. By Lemma 2.3.9, then, $R_{f_n}(v) + R_{f_n}(w_1) = R_{f_1}(v) + R_{f_1}(w_1)$. It follws that R is weakly consistent. $\square$

Definition 2.3.11. If $\Gamma = \{v_1, ..., v_n\}$ is a simple closed edge path in K, and if $A > 0$, then a face label R for K is $\Gamma(A)$ means that if $f_1, ..., f_n$ are the faces on the left side of Γ and $f'_1, ..., f'_n$ are the faces on the right side of Γ, with each pair f_i and f'_i sharing an edge $\langle v_i, v_{i+1}\rangle$ in Γ, then $R_{f'_i}(v_i) = A \cdot R_{f_i}(v_i)$ and $R_{f'_i}(v_{i+1}) = A \cdot R_{f_i}(v_{i+1})$ for $i = 1, ..., n-1$. The positive real number A is the **affine factor**.

Definition 2.3.12. A face label R for K is an **affine packing label** for K with affine factors A and B (positive real numbers) if

1. R is strongly consistent,
2. R satisfies the packing condition,
3. There is a fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$ for K such that R is $\Gamma_1(A)$ and $\Gamma_2(B)$.

2.4 Fundamental Pair Construction

It is important to establish the existence of fundamental pairs. We begin by giving an algorithm for constructing a fundamental domain $\widehat{K}$ from K. The algorithm is based on the *Drawing order meta-code* described in Stephenson (2005).

Definition 2.4.1. A **local modification at a vertex** v of a chain of faces is simply the replacement of a subchain of faces all of which belong to $F(v)$ with the other subchain of $F(v)$, keeping fixed the first and last faces of the subchain.

Fundamental Domain Construction Algorithm [FDCA]. Fix a vertex $v_0 \in K$.

1. Let $L = \widehat{K}$ be the oriented closed chain of faces forming the flower $F(v)$.
2. Create list $\mathcal{V}$ of vertices on the outer edge of L.
3. Cycle once through $v \in \mathcal{V}$.
   (a) Perform a local modification of L at v if none of the replacement faces is already in L.
   (b) Otherwise, there will be a face $f \in F(v)$ which is in neither $\widehat{K}$ nor in L, but which has a contiguous face g in L. Modify L as follows:

$$\{\dots, g, \dots\} \to \{\dots, f, g, f, \dots\}.$$

(c) Modify $\widehat{K}$ by adding to it any new faces in L (an added face shares an edge ONLY with the face before and the face after it in the list L).

4. If L has changed, return to step 2.

5. Otherwise, $\widehat{K}$ is a combinatorial fundamental domain for K.

Notation. Let V and $\widehat{V}$ be the number of vertices of K and $\widehat{K}$ respectively. Similarly for E, $\widehat{E}$, F, and $\widehat{F}$. Let V_∂ be the number of vertices of K used in the construction of the boundary of $\widehat{K}$, and let $\widehat{V}_\partial$ be the number of boundary vertices of $\widehat{K}$. Similarly, let $V_\circ$ be the number of vertices of K used in the construction of the interior of $\widehat{K}$, and let $\widehat{V}_\circ$ be the number of interior vertices of $\widehat{K}$. Similarly for E_∂, $\widehat{E}_\partial$, $E_\circ$, $\widehat{E}_\circ$, and F and$\widehat{F}$.

Lemma 2.4.2. *The result of each iteration of step 3 of the FDCA is a combinatorial closed disc.*

Proof. In step 1 $\widehat{K}$ is the flower $F(v_0)$, and hence is a combinatorial closed disc. The modification to L in step 3(a) or in step 3(b) and the resulting modification to $\widehat{K}$ in step 3(c) preserve this property. □

Remark. It is clear that $\widehat{V} = \widehat{V}_\partial + \widehat{V}_\circ$ and that $\widehat{E} = \widehat{E}_\partial + \widehat{E}_\circ$.

Lemma 2.4.3. $F = \widehat{F}$.

Proof. Since K is connected, each face of F will be used at least once in the construction of $\widehat{K}$, so $F \leq \widehat{F}$. And faces of K already used in the construction of $\widehat{K}$ are by construction in step 3 not used again, so $\widehat{F} \leq F$. □

Lemma 2.4.4. $V = V_\partial + V_\circ$.

Proof. Since each face of K is used in the construction of $\widehat{K}$, each vertex of K is also used. Hence, $V \leq V_\partial + V_\circ$. If a vertex $v \in K$ is used in the construction of an interior vertex $w \in \widehat{K}$, then all faces of $F(v)$ are attached to w as described in step 1 or in steps 3(a) and 3(c). Suppose that v is also used in the construction of a boundary vertex $w' \in \widehat{K}$. At least one face $f \in F(v)$ must be used to construct a face of $F(w')$. Hence, f is used twice in the construction of $\widehat{K}$, contradicting the previous lemma. It follows that a vertex of K cannot be used in the construction of both an interior and a boundary vertex of $\widehat{K}$. So $V_\partial + V_\circ \leq V$. □

Lemma 2.4.5. $V_\circ = \widehat{V}_\circ$.

Proof. By definition, $V_\circ \leq \widehat{V}_\circ$. As in the proof of Lemma 2.4.4, if a vertex of K is used to construct more than one interior vertex of $\widehat{K}$, it would follow that a face of $F(v)$ would be used to construct more than one face of $\widehat{K}$. So $\widehat{V}_\circ \leq V_\circ$. □

Lemma 2.4.6. $E = E_\partial + E_\circ$.

Proof. Similar to the proof of Lemma 2.4.4. □

Lemma 2.4.7. $E_\circ = \widehat{E}_\circ$.

Proof. Similar to the proof of Lemma 2.4.5. □

Lemma 2.4.8. $2E_\partial = \widehat{E}_\partial$.

Proof. Let e be an edge of K used to construct a boundary edge d of $\widehat{K}$. Since K has no boundary, e is shared by exactly two faces f and g of K. Since e is used to construct a boundary edge of d, one of f or g gets used to construct the single face attached to d in $\widehat{K}$. It follows that when the other of f or g gets used in the construction of $\widehat{K}$ it must be used in such a way that e is used to construct a boundary edge of $\widehat{K}$ (else f or g would be used more than once in the construction). So e is used exactly twice in the construction of $\widehat{K}$. □

Lemma 2.4.9. $\widehat{E}_\partial = \widehat{V}_\partial$.

Proof. In step 1 of the FDCA, the boundary of $\widehat{K}$ is a simple closed edge path. The modification to L in step 3(a) or in step 3(b) and the resulting modification to $\widehat{K}$ in step 3(c) preserve this property. And a simple closed edge path has the same number of vertices as edges. □

Proposition 2.4.10. $\widehat{V}_\partial = 2 + 2V_\partial$.

Proof. The Euler characteristics of $\widehat{K}$ and K are one and zero, respectively. By the above lemmas, then,

$$\begin{aligned}
\chi(\widehat{K}) - \chi(K) &= 1 - 0, \\
\widehat{V} - \widehat{E} + \widehat{F} - (V - E + F) &= 1, \\
\widehat{V} - V - \widehat{E} + E &= 1, \\
\widehat{V}_\circ + \widehat{V}_\partial - V_\circ - V_\partial - \widehat{E}_\circ - \widehat{E}_\partial + E_\circ + E_\partial &= 1, \\
\widehat{V}_\partial - V_\partial - \widehat{E}_\partial + E_\partial &= 1, \\
2\widehat{V}_\partial - 2V_\partial - 2\widehat{E}_\partial + \widehat{E}_\partial &= 2, \\
2\widehat{V}_\partial - 2V_\partial - \widehat{E}_\partial &= 2, \\
2\widehat{V}_\partial - 2V_\partial - \widehat{V}_\partial &= 2, \\
\widehat{V}_\partial - 2V_\partial &= 2.
\end{aligned}$$

□

The main result of this section is the following:

Theorem 2.4.11. *If K is a combinatorial torus, then there exists a fundamental pair for K.*

Proof. Since vertices of K used in the construction of $\partial\widehat{K}$ must be used at least twice, it follows from Proposition 2.4.10 that there are exactly two cases. Case 1: one vertex of K used in the construction of $\partial\widehat{K}$ is used four times and all other vertices of K used in the construction of $\partial\widehat{K}$ are used twice. Case 2: two vertices of K used in the construction of $\partial\widehat{K}$ are used three times and all other vertices of K used in the construction of $\partial\widehat{K}$ are used twice. First, we claim that Case 2 can be reduced to Case 1. Suppose Case 2 occurs, and let $v, w \in K$ be the vertices used three times each. Note that in the edge path $\partial\widehat{K}$ the vertices of $\partial\widehat{K}$ constructed from v must occur alternately with those constructed from w. This follows from the algrorithm by considerations of orientation. Let $a_1, a_2, a_3 \in \partial\widehat{K}$ be the vertices constructed from v, and let $b_1, b_2, b_3 \in \partial\widehat{K}$ be the vertices constructed from w, with these vertices occuring in $\partial\widehat{K}$ as follows:

$$\{a_1, \ldots, b_1, \ldots, a_2, \ldots, b_2, \ldots, a_3, \ldots, b_3, \ldots\}.$$

Let γ_{11} be the boundary edge path in $\widehat{K}$ joining a_1 and b_1 and not containing any of the other a_i or b_i. Similarly, let γ_{12} join b_1 and a_2. And similarly for γ_{22}, γ_{23}, γ_{33}, and γ_{31}. By the connectedness of $\widehat{K}$, we may choose an interior edge path σ connecting

b_1 and b_2. Perform a combinatorial cut along σ, and then paste γ_{12} to γ_{33}. The result is that a_2 is eliminated from the boundary and b_2 is duplicated along the boundary, so that Case 1 now occurs.

In Case 1, let $v \in K$ be the vertex used four times, and let $a_1, a_2, a_3, a_4 \in \partial\widehat{K}$ be the vertices constructed from v. As above, let γ_{12}, γ_{23}, γ_{34}, and γ_{41} be the disjoint edge paths of $\partial\widehat{K}$ with endpoints the a_i's. Let Γ_{12} and Γ_{23} be the corresponding edge paths of K. Then $\Gamma = \Gamma_{12} * \Gamma_{23}$ is a fundamental pair for K with corner v. $\square$

2.5 Face Labels

Definition 2.5.1. Let $\langle u, v\rangle$ be an edge of K and let f and g be the faces of K sharing this edge. Let S be a face label for K. An S-**multiplication factor** for edge $\langle u, v\rangle$ at u is:

$$MS^f_g(u) := \frac{S_f(u)}{S_g(u)}.$$

Lemma 2.5.2. *Let $\langle u, v\rangle$ be an edge of K and let f and g be the faces of K sharing this edge. Then a face label S is strongly consistent at the edge $\langle u, v\rangle$ if and only if $MS^f_g(u) = MS^f_g(v)$.*

Proof. Immediate from the definition of strong consistency. $\square$

Notation. If the face label S is strongly consistent, and if $\langle u, v\rangle$ is an edge of K shared by faces f and g, let

$$MS^f_g := MS^f_g(u) = MS^f_g(v).$$

Definition 2.5.3. Let S and T be face labels for K. Then $S \sim T$ if there exist positive numbers α_f for each face f in K such that $\alpha_f \cdot S_f(v) = T_f(v)$ for all vertices v in the face f.

Lemma 2.5.4. *The relation $\sim$ is an equivalence relation on the set of face labels for K.*

Lemma 2.5.5. *If S is a strongly consistent face label and if $S \sim T$, then T is strongly consistent.*

Proposition 2.5.6. *Let K be a combinatorial torus, $\Gamma = \Gamma_1 * \Gamma_2$ a fundamental pair for K, and T a strongly consistent face label for K. Then there exists a face label S*

for K such that $S \sim T$ and $MS_g^f = 1$ for each pair of faces f and g sharing an edge not belonging to Γ.

Proposition 2.5.7. *If S and S' are face labels satisfying the conclusion of Proposition 2.5.6, then there is a positive number β such that $\beta \cdot S_f(v) = S'_f(v)$ for all vertices $v \in K$ and faces $f \in F(v)$.*

Proposition 2.5.8. *There exist positive numbers A and B such that if S is a face label satisfying the conclusion of Proposition 2.5.6, then S is $\Gamma_1(A)$ and $\Gamma_2(B)$.*

2.6 The Face Label $S(\Gamma, A, B)$

Let K be a combinatorial torus, $\Gamma = \Gamma_1 * \Gamma_2$ a fundamental pair for K, and A and B positive numbers. In this secton we define a particular face label S that is strongly consistent, $\Gamma_1(A)$, and $\Gamma_2(B)$. This face label will play an important role in Chapter 3.

Remark. It is clear that each face flower $F(v)$ inherits an orientation from its faces.

Definition 2.6.1. If Γ is a simple edge path with successive vertices u_{i-1}, u_i, u_{i+1} and faces $h_i = \langle u_i, u_{i+1}, t_i \rangle$ and $h_{i-1} = \langle u_{i-1}, u_i, t_{i-1} \rangle$ to the left of Γ, then $F_L(u_i)$, the **faces in the flower of u_i to the left of Γ**, is the collection of faces ordered with respect to the orientation of $F(u_i)$ beginning with h_i and ending with h_{i-1}. Similarly, if $h'_i = \langle u_{i+1}, u_i, t'_i \rangle$ and $h'_{i-1} = \left\langle u_i, u_{i-1}, t'_{i-1} \right\rangle$ are faces to the right of Γ, then $F_R(u_i)$, the **faces in the flower of u_i to the right of Γ**, is the collection of faces ordered with respect to the orientation of $F(u_i)$ beginning with h'_{i-1} and ending with h'_i.

Let $\Gamma_1 = \{v_1, ..., v_n\}$, $\Gamma_2 = \{w_1, ..., w_m\}$, and let

$$\begin{aligned}
f_i &= \langle v_i, v_{i+1}, u_i \rangle \quad i = 1, ..., n-1 \quad \text{faces on the left side of } \Gamma_1,\\
f'_i &= \langle v_{i+1}, v_i, u'_i \rangle \quad i = 1, ..., n-1 \quad \text{faces on the right side of } \Gamma_1,\\
g_i &= \langle w_i, w_{i+1}, t_i \rangle \quad i = 1, ..., m-1 \quad \text{faces on the left side of } \Gamma_2,\\
g'_i &= \langle w_{i+1}, w_i, t'_i \rangle \quad i = 1, ..., m-1 \quad \text{faces on the right side of } \Gamma_2.
\end{aligned}$$

Let $v := v_1 = v_n = w_1 = w_m$ be the corner of Γ. Since we have edge paths Γ_1 and Γ_2 to consider, let $F_{Li}(u)$ be the collection of faces of $F(u)$ to the left of Γ_i, where $u \in \Gamma_i$. Similarly for $F_{Ri}(u)$. Finally, for the corner vertex v, let $F_{LR}(v) := F_{L1}(v) \cap F_{R2}(v)$, and similarly for $F_{RL}(v)$, $F_{LL}(v)$, $F_{RR}(v)$.

Without loss of generality, let $g_{m-1} \in F_{R1}(v)$. It follows from considerations such as those in the proof of Theorem 2.4.11 that $g_1 \in F_{L1}(v)$ (otherwise a combinatorial cut along Γ would fail to produce a combinatorial closed disc). By considering the orientation of $F(v)$, it also follows that $f_1 \in F_{R2}(v)$ and $f_{n-1} \in F_{L2}(v)$.

Observe that:

$$\begin{gathered} g_1 = \langle w_1, w_2, t_1 \rangle = \langle v, w_2, t_1 \rangle \,, \\ f_{n-1} = \langle v_{n-1}, v_n, u_{n-1} \rangle = \langle v_{n-1}, v, u_{n-1} \rangle = \langle v, u_{n-1}, v_{n-1} \rangle \,. \end{gathered}$$

Since $g_1, f_{n-1} \in F_{LL}(v)$ it is possible that $w_2 = u_{n-1}$. In that case g_1 and f_{n-1} would share the oriented edge $\langle v, w_2 \rangle = \langle v, u_{n-1} \rangle$ and hence would be identical faces. Similarly, it is possible that $g_{m-1} = f'_{n-1}$, $f'_1 = g'_{m-1}$, and that $g'_1 = f_1$. Observe also that successive faces on the same side of an edge path might be identical. For example,

$$\begin{gathered} f_j = \langle v_j, v_{j+1}, u_j \rangle \,, \\ f_{j+1} = \langle v_{j+1}, v_{j+2}, u_{j+1} \rangle = \langle u_{j+1}, v_{j+1}, v_{j+2} \rangle \,. \end{gathered}$$

So if $u_{j+1} = v_j$, then f_j and f_{j+1} share an oriented edge and therefore must be identical. These observations show that we must be careful to make sure that our face label is well defined. We do this by specifying the values of S at vertices and using disjoint collections of faces contained in the face flowers of these vertices.

Definition 2.6.2. Let $A, B > 0$. Define a face label $S := S(\Gamma, A, B)$ as follows:

$$S_f(u) = \begin{cases} 1 & \text{if } u \notin \Gamma,\ f \in F(u), \\ 1 & \text{if } u \in \Gamma_1,\ u \neq v,\ f \in F_{L1}(u), \\ A & \text{if } u \in \Gamma_1,\ u \neq v,\ f \in F_{R1}(u), \\ 1 & \text{if } u \in \Gamma_2,\ u \neq v,\ f \in F_{L2}(u), \\ B & \text{if } u \in \Gamma_2,\ u \neq v,\ f \in F_{R2}(u), \\ 1 & \text{if } u = v,\ f \in F_{LL}(v), \\ A & \text{if } u = v,\ f \in F_{RL}(v), \\ B & \text{if } u = v,\ f \in F_{LR}(v), \\ AB & \text{if } u = v,\ f \in F_{RR}(v). \end{cases}$$

See Figure 2.3 for an example of S.

Proposition 2.6.3. *The face label $S = S(\Gamma, A, B)$ is strongly consistent.*

Proof. It is easy to check that S satisfies:

$$\begin{aligned}
&u, v \text{ not both in the same } \Gamma_i \Rightarrow MS_g^f(u) = MS_g^f(v) = 1, \\
&u, v \in \Gamma_1 \Rightarrow MS_g^f(u) = MS_g^f(v) \in \left\{A, \frac{1}{A}\right\}, \\
&u, v \in \Gamma_2 \Rightarrow MS_g^f(u) = MS_g^f(v) \in \left\{B, \frac{1}{B}\right\}.
\end{aligned}$$

for any edge $\langle u, v\rangle$ of K, with f and g the faces of K sharing this edge. By Lemma 2.5.2 it follows that S is strongly consistent. This completes the proof of the proposition. □

Proposition 2.6.4. *The face label $S = S(\Gamma, A, B)$ is $\Gamma_1(A)$ and $\Gamma_2(B)$.*

Proof. Another routine checking of cases. For example, since $f_1 \in F_{R2}$ it must also be the case that $f_1' \in F_{R2}$. Otherwise by orientation, $\langle v, v_2, u_1\rangle = f_1 = g'_{m-1} = \langle v, w_{m-1}, t_{m-1}\rangle$ and hence $w_{m-1} = v_2$, contradicting that v is the only vertex shared by Γ_1 and Γ_2. By definition, $f_1 \in F_{L1}$ and $f_1' \in F_{R1}$. So $f_1 \in F_{LR}$ and $f_1' \in F_{RR}$ and hence $S_{f_1'}(v) = AB = A \cdot S_{f_1}(v)$. □

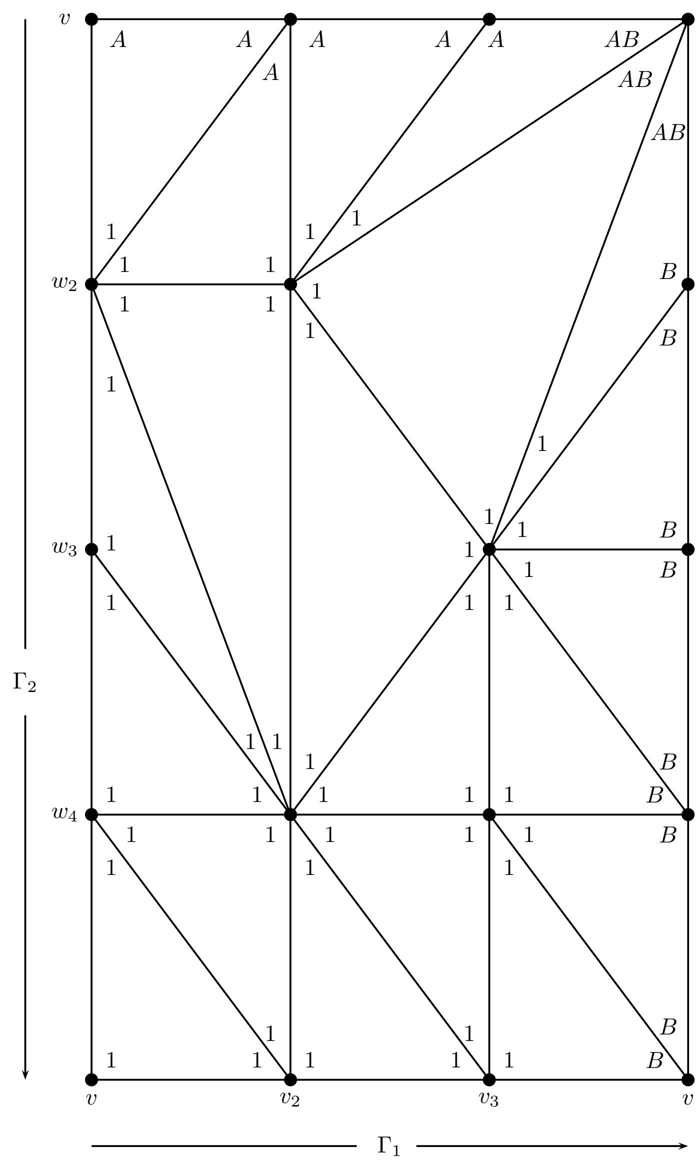

Figure 2.3: Face Label $S = S(\Gamma, A, B)$

Chapter 3

Affine Packing Labels for Combinatorial Tori

3.1 A New Proof: Existence of Vertex Packing Labels

In Stephenson (2005), the existence of euclidean vertex packing labels for combinatorial tori is proven by constructing a combinatorial covering complex and appealing to uniformization results in the theory of Riemann surfaces. We present a new existence proof that appeals only to counting results following from the value of the Euler characteristic for a combinatorial torus and to two elementary circle packing results (monotonicity and continuity of angle sums). A straightforward generalization of this new proof will be used to establish the existence of affine packing labels for combinatorial tori in Section 3.3.

Monotonicity of Angle Sums. If vertices v and v' are neighbors in the complex K, and if R_1 and R_2 are euclidean labels for K such that $R_1(v) < R_2(v)$ and $R_1(w) = R_2(w)$ for all other vertices $w \in K$, then:

$$\theta_{R_1}(v) > \theta_{R_2}(v),$$
$$\theta_{R_1}(v') < \theta_{R_2}(v').$$

Continuity of Angle Sums. Let $v_1 \dots v_n$ be the vertices neighoring v in the complex K, and let

$$\theta : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$$

be the function assigning to $(r_1, \ldots, r_n, r_{n+1})$ the angle sum at v with respect to any euclidean label R for K such that $R(v_i) = r_i$ $(i = 1 \ldots n)$ and $R(v) = r_{n+1}$. The function θ is continuous.

Counting. Let K be a combinatorial torus. Then its Euler characteristic is zero:

$$0 = \chi(K) = V - E + F.$$

Since K has no boundary, $3F = 2E$. It follows that $F = 2V$.

Theorem 3.1.1. *If K is a combinatorial torus, then there exists a euclidean vertex packing label for K.*

Proof. Let $v_0 \in K$ be fixed. Define the label R_0 by $R_0(v) = 1$ for all $v \in K$. Define the sets

$$\mathcal{S} = \{v \in K \backslash \{v_0\} : \theta_{R_0}(v) \leq 2\pi\},$$

and

$$\Phi = \{R : \theta_R(v) \leq 2\pi \text{ for } v \in \mathcal{S}, R(v) = R_0(v) \text{ for } v \in K \backslash (\mathcal{S} \cup \{v_0\}), R(v_0) = 1\}.$$

Observe that Φ is nonempty since it contains R_0. So it makes sense to let $\widehat{R}$ be defined by:

$$\widehat{R}(v) = \inf_{R \in \Phi} R(v)$$

for all $v \in K$. We claim that $\widehat{R}$ is a genuine label. That is, it is nondegenerate.

Proposition 3.1.2. *$\widehat{R}(v) > 0$ for all $v \in K$.*

Proof. Suppose not. Define the sets

$$K_0 = \{v \in K : \widehat{R}(v) = 0\},$$
$$K_\infty = \{v \in K : \widehat{R}(v) > 0\}.$$

Lemma 3.1.3. *$K_0 \subset \mathcal{S}$.*

Proof. If $w \in K \backslash \mathcal{S}$, then $w = v_0$ or $\theta_{R_0}(w) > 2\pi$. In either case, by definition of Φ, $R(w) = R_0(w) > 0$ for all labels $R \in \Phi$. It follows that $\widehat{R}(w) > 0$, and hence that $w \notin K_0$. This completes the proof of the lemma. $\square$

Observe that $V_0 = |K_0| > 0$ by supposition and that $V_\infty = |K_\infty| > 0$ since $v_0 \in K_\infty$.

It follows from monotonicity of angle sums that Φ is closed under minima. That is, if R_1 and R_2 are labels in Φ and if R_{min} is the label defined by $R_{min}(v) = \min\{R_1(v), R_2(v)\}$ for all $v \in K$, then R_{min} is also in Φ. And since Φ is closed under minima, we may choose a sequence of labels $R_n \in \Phi$ such that $R_n(v) \to \widehat{R}(v)$ as $n \to \infty$ for all $v \in K$.

Let $\alpha_n(u; v, w)$ be the angle relative to the label R_n at the vertex u in the face $\langle u, v, w\rangle$. The following two lemmas are consequences of the euclidean law of cosines.

Lemma 3.1.4. *Let $\langle u, v, w\rangle$ be a face of K. If $u \in K_\infty$ and $v, w \in K_0$, then $\alpha_n(u; v, w) \to 0$ and $\alpha_n(v; w, u) + \alpha_n(w; u, v) \to \pi$ as $n \to \infty$.*

Lemma 3.1.5. *Let $\langle u, v, w\rangle$ be a face of K. If $u, v \in K_\infty$ and $w \in K_0$, then $\alpha_n(u; v, w) + \alpha_n(v; w, u) \to 0$ and $\alpha_n(w; u, v) \to \pi$ as $n \to \infty$.*

Let $K_{(n,m)}$ be the set of faces of K having n vertices in K_0 and m vertices in K_∞, where n and m are nonnegative integers satisfying $n + m = 3$. Let $F_{(n,m)} = |K_{(n,m)}|$, and let $F_0 = F_{(3,0)}$, $F_\infty = F_{(0,3)}$, and $F_m = F_{(2,1)} + F_{(1,2)}$ (the number of faces having "mixed" vertices).

Lemma 3.1.6. $\sum_{v\in K_0} \theta_{R_n}(v) \to \pi(F_0 + F_m)$ *as $n \to \infty$.*

Proof. For a face $f = \langle u, v, w\rangle$ of K, let $\beta_n(f)$ be the sum of the angles relative to the label R_n at the vertices of f that belong to K_0. Using this notation, the angles in the sum appearing in the statement of the lemma may be reorganized as:

$$\begin{aligned}\sum_{v\in K_0} \theta_{R_n}(v) &= \sum_{f\in K_{(3,0)}} \beta_n(f) + \sum_{f\in K_{(2,1)}} \beta_n(f) + \sum_{f\in K_{(1,2)}} \beta_n(f),\\ &= \pi F_0 + \sum_{f\in K_{(2,1)}} \beta_n(f) + \sum_{f\in K_{(1,2)}} \beta_n(f). \end{aligned} \tag{3.1}$$

By Lemma 3.1.4, if $f \in K_{(2,1)}$ then $\beta_n(f) \to \pi$ as $n \to \infty$. Similarly, by Lemma 3.1.5, if $f \in K_{(1,2)}$ then $\beta_n(f) \to \pi$ as $n \to \infty$. By (3.1), then,

$$\sum_{v\in K_0} \theta_{R_n}(v) \to \pi F_0 + \pi F_{(2,1)} + \pi F_{(1,2)} = \pi(F_0 + F_m)$$

as $n \to \infty$. This completes the proof of the lemma. □

Lemma 3.1.7. $2V_\infty > F_\infty$.

Proof. If we visit each face and count the vertices of K_0 in that face, we count a total of $3F_0 + 2F_{(2,1)} + F_{(1,2)}$. Similarly, if we visit each face and count the vertices of K_∞ in that face, we count a total of $3F_\infty + 2F_{(1,2)} + F_{(2,1)}$. Moreover, since by counting $F = 2V$,

$$\begin{aligned}
[3F_0 + 2F_{(2,1)} + F_{(1,2)}] + [3F_\infty + 2F_{(1,2)} + F_{(2,1)}] &= 3F_0 + 3F_{(2,1)} + 3F_{(1,2)} + 3F_\infty, \\
&= 3F, \\
&= 6V, \\
&= 6V_0 + 6V_\infty.
\end{aligned}$$

It follows that $6V_\infty = 3F_\infty + 2F_{(1,2)} + F_{(2,1)}$. Since there are vertices in K_0 and in K_∞, by connectedness of K there must be a face in K with "mixed" vertices. I.e., $F_{(1,2)} + F_{(2,1)} > 0$. It follows that $6V_\infty = 3F_\infty + 2F_{(1,2)} + F_{(2,1)} > 3F_\infty$. This completes the proof of the lemma. $\square$

Lemma 3.1.8. *$F_0 + F_m > 2V_0$.*

Proof. By Lemma 3.1.7 and by counting,

$$\begin{aligned}
2V_\infty &> F_\infty, \\
2V_\infty + 2V_0 &> F_\infty + 2V_0, \\
2V &> F_\infty + 2V_0, \\
F &> F_\infty + 2V_0, \\
F_0 + F_m + F_\infty &> F_\infty + 2V_0, \\
F_0 + F_m &> 2V_0.
\end{aligned}$$

This completes the proof of the lemma. $\square$

To complete the proof of the proposition, observe that by Lemma 3.1.6 and Lemma 3.1.8 we may choose an N so large that:

$$\sum_{v \in K_0} \theta_{R_N}(v) > 2\pi V_0.$$

So for some $v \in K_0$ we must have $\theta_{R_N}(v) > 2\pi$. But $v \in K_0$ implies by Lemma 3.1.3 that $v \in \mathcal{S}$. And $R_N \in \Phi$, so $\theta_{R_N}(v) \leq 2\pi$. Contradiction. $\square$

Now we return to the proof of the theorem. We have already observed that Φ is nonempty. And the label $\widehat{R}$ is nondegenerate by Proposition 3.1.2. Moreover, by continuity of angle sums, monotonicity of angle sums, and definition of $\widehat{R}$, $\theta_{\widehat{R}}(v) = 2\pi$ for all $v \in \mathcal{S}$.

Put $R_{(1)} = \widehat{R}$ and define $\mathcal{S}_1$ by replacing 'R_0' with '$R_{(1)}$' in the definition of $\mathcal{S}$. Define Φ_1 by replacing 'R_0' with '$R_{(1)}$' and '$\mathcal{S}$' with '$\mathcal{S}_1$' in the definition of Φ. Clearly $\mathcal{S} \subset \mathcal{S}_1$. If $\mathcal{S} \neq \mathcal{S}_1$ then arguing as above we can again produce a label that forces all angle sums of vertices in $\mathcal{S}_1$ to be 2π. After finitely many iterations of this process (K is finite) we reach a stage at which $\mathcal{S}_i = \mathcal{S}_{i+1}$. For ease of notation, call the label at this stage R_0.

Let $\mathcal{L} = K \backslash \{v_0\}$ and observe that $\theta_{R_0}(v) \geq 2\pi$ for all $v \in \mathcal{L}$. Define the set

$$\Theta = \{R : \theta_R(v) \geq 2\pi \text{ for } v \in \mathcal{L}, R(v_0) = 1\},$$

and define $\widetilde{R}$ as the supremum of Θ. Observe that Θ is nonempty since it contains R_0. So it makes sense to let $\widetilde{R}$ be defined by:

$$\widetilde{R}(v) = \sup_{R \in \Phi} R(v)$$

for all $v \in K$. We claim that $\widetilde{R}$ is a genuine label. That is, it is nondegenerate.

Proposition 3.1.9. *$\widetilde{R}(v) < \infty$ for all $v \in K$.*

Proof. Suppose not. Define the sets $K_0 = \{v \in K : \widetilde{R}(v) < \infty\}$ and $K_\infty = \{v \in K : \widetilde{R}(v) = \infty\}$. It is clear that $K_\infty \subset \mathcal{L}$. It follows from monotonicity of angle sums that Θ is closed under maxima. Hence we may choose a sequence of labels R_n such that $R_n(v) \to \widetilde{R}(v)$ as $n \to \infty$ for all $v \in K$. Let $\gamma_n(f)$ be the sum of the angles relative to the label R_n at the vertices of f that belong to K_∞. The proofs of the following three lemmas are similar to proofs of lemmas used above.

Lemma 3.1.10. *Let $\langle u, v, w\rangle$ be a face of K. If $u \in K_\infty$ and $v \in K_0$, then $\alpha_n(u; v, w) \to 0$ as $n \to \infty$.*

Lemma 3.1.11.

$$\begin{aligned}
\sum_{v\in K_\infty} \theta_{R_n}(v) &= \sum_{f\in K_{(0,3)}} \gamma_n(f) + \sum_{f\in K_{(1,2)}} \gamma_n(f) + \sum_{f\in K_{(2,1)}} \gamma_n(f), \\
&= \pi F_\infty + \sum_{f\in K_{(1,2)}} \gamma_n(f) + \sum_{f\in K_{(2,1)}} \gamma_n(f), \\
&\to \pi F_\infty \ \text{as}\ n \to \infty.
\end{aligned}$$

Lemma 3.1.12. $2V_\infty > F_\infty$.

To complete the proof of the proposition, observe that by Lemma 3.1.11 and Lemma 3.1.12 we may choose an N so large that:

$$\sum_{v\in K_\infty} \theta_{R_N}(v) < 2\pi V_\infty.$$

So for some $v \in K_\infty$ we must have $\theta_{R_N}(v) < 2\pi$. But $v \in K_\infty \Rightarrow v \in \mathcal{L}$, and $R_N \in \Theta$, so $\theta_{R_N}(v) \geq 2\pi$. Contradiction. □

We now complete the proof of the theorem. We have already observed that Θ is nonempty. And the label $\widetilde{R}$ is nondegenerate by 3.1.9. Moreover, by continuity of angle sums, monotonicity of angle sums, and definition of $\widetilde{R}$, $\theta_{\widetilde{R}}(v) = 2\pi$ for all $v \in \mathcal{L}$. It remains to observe that the total angle sum of any label R for K is $\pi F = 2\pi V$. So

$$\begin{aligned}
\theta_{\widetilde{R}}(v_0) &= \sum_{v\in K} \theta_{\widetilde{R}}(v) - \sum_{v\in\mathcal{L}} \theta_{\widetilde{R}}(v), \\
&= 2\pi V - 2\pi(V-1), \\
&= 2\pi.
\end{aligned}$$

So $\widetilde{R}$ is a euclidean packing label for K. □

3.2 Discussion

The following is an outline of the process described in the proof of Theorem 3.1.1, with the arguments concerning nondegeneracy of labels suppressed. K is a combinatorial torus and v_0 is a fixed vertex of K.

1. Begin with vertex label R_0 such that $R_0(v) = 1$ for all vertices $v \in K$.

2. Let $\mathcal{S}$ be the set of vertices of $K\backslash\{v_0\}$ having R_0-angle sums $\leq 2\pi$.

3. Obtain the label R_1 by "decreasing" the R_0-radii of vertices in $\mathcal{S}$ (isolated vertices of $\mathcal{S}$ having angle sums 2π would not have their radii adjusted) so that all vertices of $\mathcal{S}$ have R_1-angle sums 2π.

4. All radii adjustments in the previous step were decreases to the R_0-radii of vertices in $\mathcal{S}$.

   - So by monotonicity of angle sums any vertex $w \in K\backslash(\mathcal{S} \cup \{v_0\})$ whose angle sum is affected by these adjustments would have a decrease in angle sum.
   - Since the R_0-angle sum of w was greater than 2π, it is possible that after a decrease in angle sum the R_1-angle sum of w would $\leq 2\pi$.
   - It follows that the set of vertices $\mathcal{S}_1$ in $K\backslash\{v_0\}$ having R_1-angle sums $\leq 2\pi$ contains the set of vertices $\mathcal{S}$ having R_0-angle sums $\leq 2\pi$.

5. Now repeat the process: "decrease" the R_1-radii of vertices in $\mathcal{S}_1$ to obtain a label R_2 such that all vertices of $\mathcal{S}_1$ have R_2-angle sums 2π. The set of vertices $\mathcal{S}_2$ in $K\backslash\{v_0\}$ having R_2-angle sums $\leq 2\pi$ contains the set of vertices $\mathcal{S}_1$ having R_1-angle sums $\leq 2\pi$.

6. Repeat the process until obtaining a set $\mathcal{S}_{i+1} = \mathcal{S}_i$ (this must happen since K is finite and the sets $\mathcal{S}_n$ are an increasing sequence of subsets of K).

   - It follows that a vertex $w \in K\backslash\{v_0\}$ has R_{i+1}-angle sum $\leq 2\pi$ if and only if it has R_{i+1}-angle sum $= 2\pi$.
   - So all vertices $w \in K\backslash\{v_0\}$ have R_{i+1}-angle sums $\geq 2\pi$.
   - At this stage all excess angle occurs at vertices in $K\backslash\{v_0\}$ and all angle deficit is at v_0.

7. Obtain a final label R by increasing the R_{i+1}-radii of vertices in $K\backslash\{v_0\}$ so that their R-angle sums are 2π.

8. By counting, the total angle sum added over all vertices of K is $2\pi V$ for any label. Since the R-angle sums of the $V-1$ vertices in $K\backslash\{v_0\}$ are all 2π, it follows that the R-angle sum of v_0 must also be 2π. So R is a packing label.

Observe that throughout the adjustment process the radius of the vertex v_0 is never modified. So it could be chosen as any radius initially and the final label would preserve that choice. Observe also that we obtain a packing label by starting with an initial arbitrary label, decreasing radii of finitely many increasing sets of vertices, and then increasing radii. These phenomena reflect the fact that there is a special connection between euclidean labels and combinatorial tori.

Proposition 3.2.1. *Let K be a closed combinatorial surface (that is, a finite complex without boundary) such that $\chi(K) \neq 0$. Then there is no euclidean vertex packing label for K.*

Proof. Suppose R is a euclidean packing label for K. Since R is euclidean, the total angle (calculated face-by-face) of R is πF. Since R is a packing label, and since all vertices of K are interior vertices, the total angle (calculated by angle sums at vertices) is $2\pi V$. So $F = 2V$.

But K has no boundary, so $3F = 2E$. Hence $V - E + F = V - F/2$. It follows that $0 \neq \chi(K) = V - F/2$, and hence $F \neq 2V$. Contradiction. □

3.3 Existence of Affine Packing Labels

Definition 3.3.1. If R is a vertex label and S is a face label, let the product $R \cdot S$ be the face label given by:

$$(R \cdot S)_f(v) = R(v) \cdot S_f(v)$$

for all vertices v and faces f containing v.

Theorem 3.3.2. *If K is a combinatorial torus and S is a face label for K, then there is a vertex label R for K such that the face label $R \cdot S$ satisfies the packing condition.*

Proof. Reinterpret angles α_R and angle sums θ_R in the proof of Theorem 3.1.1 as being relative to the face label $R \cdot S$. Observe concerning this reinterpretation:

- By definition, $0 < S_f(v) < \infty$ for all vertices v and for all faces f containing v. So the vertex label R is nondegenerate if and only if the face label $R \cdot S$ is nondegenerate.

- For vertex labels R and R':

$$R(v) < R'(v) \Leftrightarrow R(v) \cdot S_f(v) < R'(v) \cdot S_f(v)$$

for all vertices v and all faces f containing v. It follows that the monotonicity of angle sums in the relationship between R and α_R that was crucial in the proof of 3.1.1 holds between R and $\alpha_{R \cdot S}$ under the reinterpretation.

- Continuity of the angle sums of $R \cdot S$ in the entries of R follows from continuity of angle sums with respect to vertex labels and continuity of multiplication.
- The counting arguments used in the proof of Theorem 3.1.1 follow from purely combinatorial considerations.

These observations are sufficient to conclude that the reinterpretation of the proof of Theorem 3.1.1 under consideration is valid and guarantees the existence of the desired vertex label. □

Lemma 3.3.3. *The face label properties strong consistency, $\Gamma_1(A)$, and $\Gamma_2(B)$ are preserved under multiplication by vertex labels.*

Proof. Immediate from the definitions. For example, strong consistency is preserved since:

$$\frac{S_f(u)}{S_f(v)} = \frac{S_g(u)}{S_g(v)} \Leftrightarrow \frac{R(u)S_f(u)}{R(v)S_f(v)} = \frac{R(u)S_g(u)}{R(v)S_g(v)}.$$

□

Theorem 3.3.4. *For any combinatorial torus K, any choice of a fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$ for K, and any choice of positive affine factors A and B, there exists an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$.*

Proof. By Theorem 3.3.2 and Lemma 3.3.3 it is enough to show that there is a face label S for K that is strongly consistent, $\Gamma_1(A)$, and $\Gamma_2(B)$. But the face label $S = S(\Gamma, A, B)$ constructed in Section 2.6 satisfies these conditions. □

3.4 A New Proof: Uniqueness of Vertex Packing Labels

In Stephenson (2005), uniqueness of euclidean packing labels for combinatorial tori is proven by appealing to uniformization results in the theory of Riemann surfaces. We present a new uniqueness proof that appeals only to monotonicity of angle sums and to counting results following from the value of the Euler characteristic for a combinatorial torus. A straightforward generalization of this new proof will be used to establish the uniqueness of affine packing labels for combinatorial tori in Section 3.5.

Definition 3.4.1. If R is a label and if $t > 0$, then $t \cdot R$ is the label given by $(t \cdot R)(v) = t \cdot R(v)$ for all $v \in K$.

Lemma 3.4.2. *If R is a euclidean packing label for the combinatorial torus K and if $t > 0$, then $t \cdot R$ is a euclidean packing label for K.*

Proof. Immediate from the euclidean law of cosines used to calculate angles and angle sums for euclidean labels. $\square$

Theorem 3.4.3. *If R and S are euclidean packing labels for the combinatorial torus K, then there is a $t > 0$ such that $R = t \cdot S$.*

Proof. Suppose not. Then by Lemma 3.4.2 we may suppose that R and S are euclidean packing labels for K such that $R \neq S$ and that there is a $v_0 \in K$ such that $R(v_0) = S(v_0)$. Since $R \neq S$, there is a $v_1 \in K$ such that $R(v_1) \neq S(v_1)$. Without loss of generality, we may suppose that $S(v_1) < R(v_1)$. Define the sets

$$K_0 = \{v \in K : R(v) \leq S(v)\},$$
$$K_1 = \{v \in K : S(v) < R(v)\}.$$

Observe that $K_0 \cup K_1 = K$, $K_0 \cap K_1 = \emptyset$, $v_0 \in K_0$, and $v_1 \in K_1$. By the connectedness of K, then, there must exist neighboring vertices $w_0 \in K_0$ and $w_1 \in K_1$.

Let m be the label defined by $m(v) = \min\{R(v), S(v)\}$ for all $v \in K$.

Lemma 3.4.4. $\theta_m(v) \leq 2\pi$ *for all* $v \in K$.

Proof. Let $v \in K$ and suppose without loss of generality that $m(v) = R(v)$. By definition of m, $m(w) \leq R(w)$ for all neighbors w of v. By monotonicity of angle sums, then, $\theta_m(v) \leq \theta_R(v)$. And since R is a packing label, $\theta_R(v) = 2\pi$. $\square$

Now since $w_0 \in K_0$, $m(w_0) = R(w_0)$. And since $w_1 \in K_1$, it follows that $m(w_1) = S(w_1) < R(w_1)$. Therefore, by definition of m, monotonicty of angle sums, and supposition:

$$\theta_m(w_0) < \theta_R(w_0) = 2\pi.$$

By this and Lemma 3.4.4,

$$\sum_{v \in K} \theta_m(v) \leq 2\pi(V - 1) + \theta_m(w_0) < 2\pi V.$$

This is a contradiction since by counting the total angle sum of any label for a combinatorial torus K must be $2\pi V$. $\square$

3.5 Uniqueness of Affine Packing Labels

Definition 3.5.1. If S is a face label and if $t > 0$, then $t \cdot S$ is the face label given by $(t \cdot S)_f(v) = t \cdot S_f(v)$ for all $v \in K$ and all $f \in F(v)$.

Lemma 3.5.2. *If S is an face label satisfying the packing condition and if $t > 0$, then the face label $t \cdot S$ satisfies the packing condition.*

Proof. Immediate from the euclidean law of cosines used to calculate angles and angle sums for euclidean face labels. □

Lemma 3.5.3. *Let K be a combinatorial torus, and let $\Gamma = \Gamma_1 * \Gamma_2$ be a fundamental pair for K. Let $A, B, t > 0$. If the face label S is $\Gamma_1(A)$ and $\Gamma_2(B)$, then the face label $t \cdot S$ is also $\Gamma_1(A)$ and $\Gamma_2(B)$.*

Proof. Immediate from the definitions. □

Theorem 3.5.4. *Let K be a combinatorial torus, and let $\Gamma = \Gamma_1 * \Gamma_2$ be a fundamental pair for K. Let $A, B > 0$. If R and S are affine packing labels that are $\Gamma_1(A)$ and $\Gamma_2(B)$, and if*

$$MS_g^f(u) = MR_g^f(u)$$

for all edges $\langle u, v\rangle$ (where f and g are the faces sharing edge $\langle u, v\rangle$), then there is a $t > 0$ such that $R = t \cdot S$.

Proof. For each vertex $v \in K$, choose a face $f_v \in F(v)$. Reinterpret the proof of Theorem 3.4.3, with '$R(v)$' understood as '$R_{f_v}(v)$' and similarly for '$S(v)$'. Also reinterpret angles and angle sums as being relative to face labels.

Since R and S have the same multiplication factors, for any $g \in F(v)$

$$R_g(v) < S_g(v) \Leftrightarrow R_{f_v}(v) < S_{f_v}(v).$$

So the monotonicity between vertex labels and angle sums that was crucial to the proof of Theorem 3.4.3 holds between the values $R_{f_v}(v)$ and angle sums. It follows that the proof of Theorem 3.4.3 is valid under the reinterpretation being considered and guarantees the desired uniqueness. □

Corollary 3.5.5. *The affine packing label shown to exist in Section 3.3 is unique up to scaling.*

Proof. The affine packing label was shown to exist by fixing a face label S and finding a vertex label R such that $R \cdot S$ is the desired affine packing label. But multiplication factors for face labels are preserved under multiplication by vertex labels, so the corollary follows from Theorem 3.5.4. □

Chapter 4

Affine Packing Labels and Circle Packings

4.1 Circle Packings for Fundamental Domains

In this section the existence of affine packing labels for combinatorial tori demonstrated in Section 3.3 is exploited to show the existence of circle packings for an associated combinatorial fundamental domain. Suppose K is a combinatorial torus and $\Gamma = \Gamma_1 * \Gamma_2$ is a fundamental pair for K. Let A and B be positive. As in Section 3.3, let $S = S(\Gamma, A, B)$, and let R be a vertex label such that $R \cdot S$ is an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$.

Let $\widehat{K}$ be the combinatorial fundamental domain obtained from K by a combinatorial cut along Γ. Modifying the notation of Section 2.4, identify $\widehat{V}_\circ$, the set of interior vertices of $\widehat{K}$, with $V_\circ$, the set of vertices of K used in the construction of $\widehat{V}_\circ$. If $\Gamma_1 = \{v_1, ..., v_n\}$ and $\Gamma_2 = \{w_1, ..., w_m\}$, then we may take $\widehat{V}_\partial = \{v_1, v_1', ..., v_n, v_n', w_1, w_1', ..., w_m, w_m'\}$ with the conventions that $v_1 = w_m$, $v_1' = w_1$, $v_n = w_m'$, and $v_n' = w_1'$. See Figure 4.1 for an example of $\widehat{K}$.

Define a vertex label $\widehat{R}$ for $\widehat{K}$ as follows:

$$\widehat{R}(u) = \begin{cases} R(u) & \text{if } u \in \widehat{V}_\circ, \\ R(u) & \text{if } u = v_i,\ i = 2, ..., n-1, \\ A \cdot R(u) & \text{if } u = v_i',\ i = 2, ..., n-1, \\ R(u) & \text{if } u = w_i,\ i = 2, ..., m-1, \\ B \cdot R(u) & \text{if } u = w_i',\ i = 2, ..., m-1, \\ R(u) & \text{if } u = v_1, \\ A \cdot R(u) & \text{if } u = v_1', \\ B \cdot R(u) & \text{if } u = v_n, \\ AB \cdot R(u) & \text{if } u = v_n'. \end{cases}$$

Remark. Since the label R is nondegenerate, the label $\widehat{R}$ is also nondegenerate.

The following lemmas are immediate consequences of the definition of $\widehat{R}$. As usual, we let $v \in K$ be the corner of Γ.

Lemma 4.1.1. *For* $u \in \widehat{V}_\circ$,

$$\widehat{R}(u) = R(u) \cdot S_f(u) \textit{ for all } f \in F(u).$$

Lemma 4.1.2. *For* $i = 2, ..., n-1$,

$$\widehat{R}(v_i) = R(v_i) \cdot S_f(v_i) \textit{ for all } f \in F_{L1}(v_i),$$
$$\widehat{R}(v_i') = R(v_i) \cdot S_f(v_i) \textit{ for all } f \in F_{R1}(v_i).$$

Lemma 4.1.3. *For* $i = 2, ..., m-1$,

$$\widehat{R}(w_i) = R(w_i) \cdot S_f(w_i) \textit{ for all } f \in F_{L2}(w_i),$$
$$\widehat{R}(w_i') = R(w_i) \cdot S_f(w_i) \textit{ for all } f \in F_{R2}(w_i).$$

Lemma 4.1.4. *The corner vertices of* $\widehat{K}$ *satisfy:*

$$\begin{aligned}
\widehat{R}(v_1) &= R(v) \cdot S_f(v) \text{ for all } f \in F_{LL}(v),\\
\widehat{R}(v'_1) &= R(v) \cdot S_f(v) \text{ for all } f \in F_{RL}(v),\\
\widehat{R}(v_n) &= R(v) \cdot S_f(v) \text{ for all } f \in F_{LR}(v),\\
\widehat{R}(v'_n) &= R(v) \cdot S_f(v) \text{ for all } f \in F_{RR}(v).
\end{aligned}$$

The next two propositions follow from the previous lemmas and the fact that $R \cdot S$ is an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$.

Proposition 4.1.5. *The angle sums for* $\widehat{R}$ *satisfy:*

$$\begin{aligned}
\theta_{\widehat{R}}(u) &= 2\pi \text{ for } u \in \widehat{V}_\circ,\\
\theta_{\widehat{R}}(v_i) + \theta_{\widehat{R}}(v'_i) &= 2\pi \text{ for } i = 2, \ldots, n-1,\\
\theta_{\widehat{R}}(w_i) + \theta_{\widehat{R}}(w'_i) &= 2\pi \text{ for } i = 2, \ldots, m-1,\\
\theta_{\widehat{R}}(v_1) + \theta_{\widehat{R}}(v'_1) + \theta_{\widehat{R}}(v_n) + \theta_{\widehat{R}}(v'_n) &= 2\pi.
\end{aligned}$$

Proposition 4.1.6. *The vetex label* $\widehat{R}$ *satisfies:*

$$\begin{aligned}
\widehat{R}(v'_i) &= A \cdot \widehat{R}(v_i) \text{ for } i = 1, \ldots, n,\\
\widehat{R}(w'_i) &= B \cdot \widehat{R}(w_i) \text{ for } i = 1, \ldots, m.
\end{aligned}$$

Proposition 4.1.7. *If* $\widehat{R}$ *and* $\widehat{R}'$ *are vertex labels for* $\widehat{K}$ *satisfying the conditions of Proposition 4.1.5 and Proposition 4.1.6, then there exists a* $t > 0$ *such that* $t \cdot \widehat{R} = \widehat{R}'$*.*

Proof. Suppose that $\widehat{R}$ and $\widehat{R}'$ are such vertex labels. Using the same notation as at the beginning of this section, define a vertex label R for K as follows:

$$R(u) = \begin{cases} \widehat{R}(u) & \text{if } u \in V_\circ,\\ \widehat{R}(u) & \text{if } u = v_i,\ i = 2, ..., n-1,\\ \widehat{R}(u) & \text{if } u = w_i,\ i = 2, ..., m-1,\\ \widehat{R}(u) & \text{if } u = v_1. \end{cases}$$

Note that using $\widehat{R}$ to define R as we have done is just the reverse of the process described at the beginning of this section. It follows that for the face label $S = S(\Gamma, A, B)$, $R \cdot S$ is an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$. Similarly, define a vertex label R' for K using $\widehat{R}'$. Note that $R' \cdot S$ is an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$. It follows by Corollary 3.5.5 that there exists a $t > 0$ such that $t \cdot R \cdot S = R' \cdot S$. Hence, $t \cdot R = R'$. By construction of R and R', then, $t \cdot \widehat{R} = \widehat{R}'$. □

Theorem 4.1.8. *There exists a circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$ in $\mathbb{C}$.*

Proof. By Proposition 4.1.5, $\widehat{R}$ is a euclidean packing label for the simply connected complex $\widehat{K}$. So the existence of P follows from the monodromy theorem: see Stephenson (2005). □

Notation. For the circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$, let z_i, z_i', y_i, and y_i' be the centers of the circles for the vertices v_i, v_i', w_i, and w_i' respectively.

Theorem 4.1.9. *There exist complex affine transformations $F(z) = \alpha z + \gamma$ and $G(z) = \beta z + \sigma$ such that $|\alpha| = A$, $|\beta| = B$, and*

$$F(z_i) = z_i' \text{ for } i = 1, ..., n,$$
$$G(y_i) = y_i' \text{ for } i = 1, ..., m.$$

Proof. Since $\widehat{R}$ is nondegenerate, $|z_2 - z_1| = \widehat{R}(v_2) + \widehat{R}(v_1) > 0$, so we may let $\alpha = (z_2' - z_1')/(z_2 - z_1)$. Observe that by Proposition 4.1.6,

$$|\alpha| = \frac{|z_2' - z_1'|}{|z_2 - z_1|} = \frac{\widehat{R}(v_2') + \widehat{R}(v_1')}{\widehat{R}(v_2) + \widehat{R}(v_1)} = \frac{A\widehat{R}(v_2) + A\widehat{R}(v_1)}{\widehat{R}(v_2) + \widehat{R}(v_1)} = A.$$

Let $\gamma = z_1' - \alpha z_1$. By definition of α and γ,

$$F(z_1) = z_1', \tag{4.1}$$
$$F(z_2) = z_2'. \tag{4.2}$$

Consider the complex numbers

$$re^{i\theta} = \frac{z_1 - z_2}{z_3 - z_2}, \tag{4.3}$$
$$r'e^{i\theta'} = \frac{z_1' - z_2'}{z_3' - z_2'}. \tag{4.4}$$

By Proposition 4.1.6,

$$\begin{aligned}
\frac{|z_1 - z_2|}{|z_3 - z_2|} &= \frac{\widehat{R}(v_1) + \widehat{R}(v_2)}{\widehat{R}(v_3) + \widehat{R}(v_2)}, \\
&= \frac{A\widehat{R}(v_1) + A\widehat{R}(v_2)}{A\widehat{R}(v_3) + A\widehat{R}(v_2)}, \\
&= \frac{\widehat{R}(v_1') + \widehat{R}(v_2')}{\widehat{R}(v_3') + \widehat{R}(v_2')}, \\
&= \frac{|z_1' - z_2'|}{|z_3' - z_2'|}.
\end{aligned}$$

So $r = r'$. By definition of angle sum and considering orientation, we may take $\theta = \theta_{\widehat{R}}(v_2)$ and $\theta' = 2\pi - \theta_{\widehat{R}}(v_2')$. It follows by Proposition 4.1.5 that $\theta = \theta'$. So the complex numbers (4.3) and (4.4) are equal, and therefore:

$$z_3 = \frac{z_3' - z_2'}{z_1' - z_2'} \cdot (z_1 - z_2) + z_2.$$

Hence, by the identities (4.1) and (4.2),

$$\begin{aligned}
\alpha z_3 + \gamma &= \frac{z_3' - z_2'}{z_1' - z_2'} \cdot (\alpha z_1 - \alpha z_2) + \alpha z_2 + \gamma, \\
&= \frac{z_3' - z_2'}{z_1' - z_2'} \cdot (z_1' - \gamma - z_2' + \gamma) + z_2', \\
&= z_3' - z_2' + z_2', \\
&= z_3'.
\end{aligned}$$

So $F(z_3) = z_3'$. This argument may now be iterated to conclude that $F(z_i) = z_i'$ for $i = 1, ..., n$.

A similar argument shows that taking $\beta = (y_2' - y_1')/(y_2 - y_1)$ and $\sigma = y_1' - \beta y_1$ yields the desired conclusions. □

Definition 4.1.10. The complex affine transformations F and G of Theorem 4.1.9 are the **side-pairing maps** for the circle packing P.

Lemma 4.1.11. *If $\widehat{R}$ and $\widehat{R}'$ are vertex labels for $\widehat{K}$ satisfying the conditions of Proposition 4.1.5 and Proposition 4.1.6, and if P and P' are circle packings such that $P \leftrightarrow \widehat{K}(\widehat{R})$ and $P' \leftrightarrow \widehat{K}(\widehat{R}')$, then there is an affine mapping of the complex plane ρ such that $\rho(P) = (P')$.*

Proof. By Proposition 4.1.7, there is a $t > 0$ such that $t \cdot R = \widehat{R}$. Since all labels are euclidean, and since angles and angle sums are preserved under scaling of euclidean labels, the result follows by reasoning similar to that in the proof of Theorem 4.1.9. $\square$

Proposition 4.1.12. *If $\widehat{R}$ and $\widehat{R}'$ are vertex labels for $\widehat{K}$ satisfying the conditions of Proposition 4.1.5 and Proposition 4.1.6, if P and P' are circle packings such that $P \leftrightarrow \widehat{K}(\widehat{R})$ (with side-pairing maps $F(z) = \alpha z + \gamma$ and $G(z) = \beta z + \sigma$) and $P' \leftrightarrow \widehat{K}(\widehat{R}')$ (with side-pairing maps $F'(z) = \alpha' z + \gamma'$ and $G'(z) = \beta' z + \sigma'$), then $\alpha = \alpha'$ and $\beta = \beta'$.*

Proof. By Lemma 4.1.11, the centers of the circles for P' are the images under an affine mapping ρ of the centers of the centers of the circle for P. Using the fact that ρ is affine, we may calculate α' as in the proof of Theorem 4.1.9:

$$\alpha' = \frac{\rho(z_2') - \rho(z_1')}{\rho(z_2) - \rho(z_1)} = \frac{z_2' - z_1'}{z_2 - z_1} = \alpha.$$

Similarly, $\beta' = \beta$. $\square$

In view of Theorem 3.3.4, Theorem 4.1.8, Proposition 4.1.7, and Theorem 4.1.12, the following definition is coherent.

Definition 4.1.13. For a combinatorial torus K, fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$, and affine factors $A, B > 0$ the associated **side-pairing parameters** α and β are the complex numbers as in Proposition 4.1.12.

Definition 4.1.14. For a combinatorial torus K, fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$, and affine factors $A, B > 0$ the moduli of the associated side-pairing parameters α and β are the **side-pairing moduli**.

Remark. Observe that by Theorem 4.1.9 the affine factors A and B are exactly the side-pairing moduli.

Figures 4.2 through 4.5 are examples of circle packings for a combinatorial fundamental domain. In Figure 4.2, the radii of circles along the top red path are the same as the radii of the corresponding circles along the bottom red path ($A = 1.0$). And the radii of the circles along the left green path are the same as the radii of the corresponding circles along the right green path ($B = 1.0$). This is a flat torus - the red paths are simply translations of one another, and similarly for the green paths.

In Figure 4.3 we begin to warp the flat torus. It is still the case that the radii of circles along the top red path are the same as the radii of the corresponding circles along the bottom red path ($A = 1.0$). But now the radii of the circles along the right green path are twice the radii of the corresponding circles along the left green path ($B = 2.0$). In Figure 4.4 the warping is intensified by keeping $A = 1.0$ and increasing B to 4.0. In Figure 4.5 the warping affects circles along both pairs of edge paths. The radii of circles along the top red path are half the radii of the corresponding circles along the bottom red path ($A = 0.5$). And the radii of the circles along the right green path are twice the radii of the corresponding circles along the left green path ($B = 2.0$).

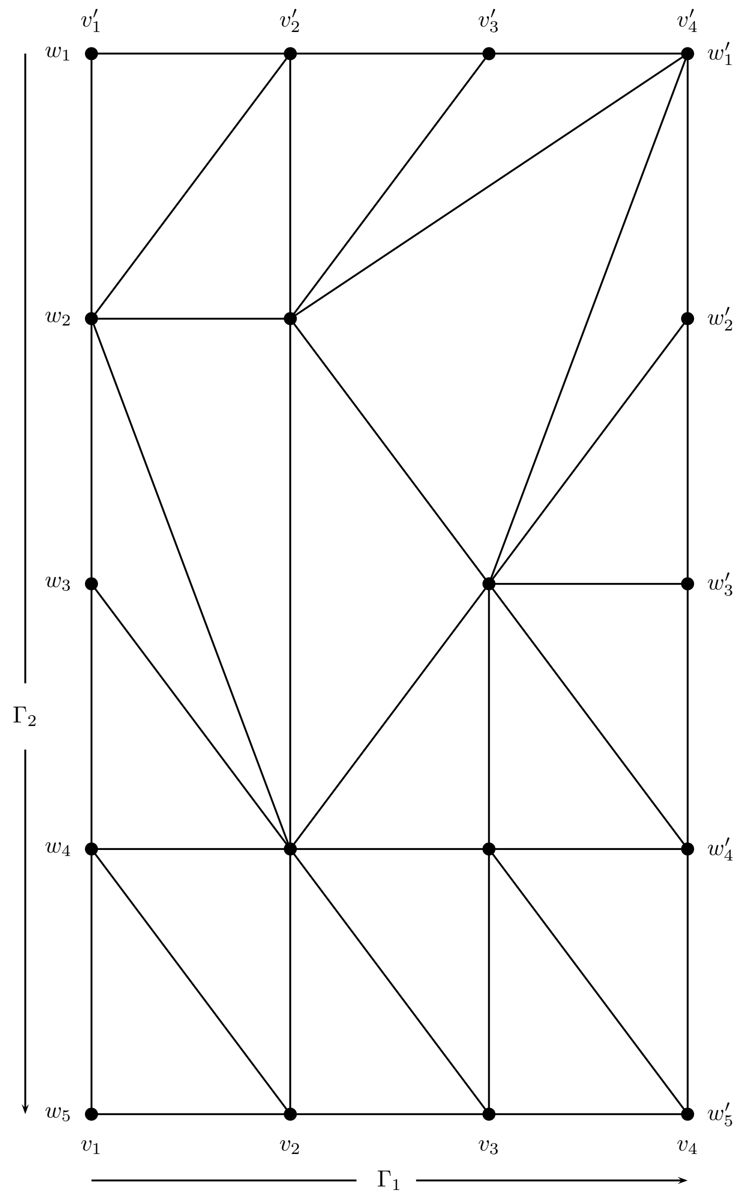

Figure 4.1: Combinatorial Fundamental Domain $\widehat{K}$

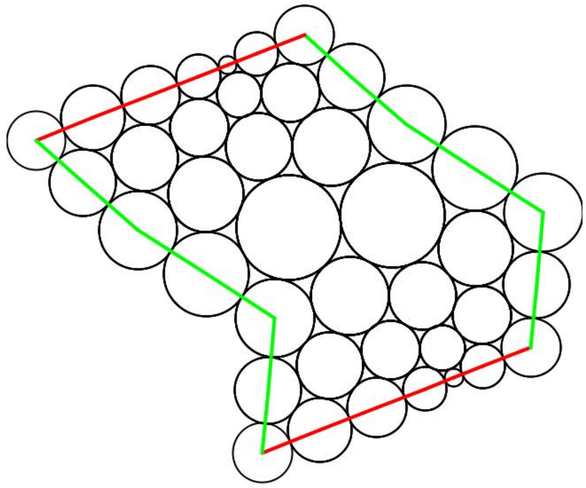

Figure 4.2: Circle Packing $P \leftrightarrow \widehat{K}(\widehat{R})$ with Affine Factors $A = 1.0$, $B = 1.0$

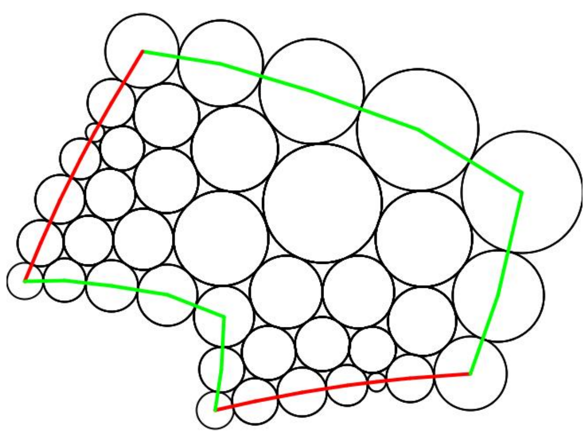

Figure 4.3: Circle Packing $P \leftrightarrow \widehat{K}(\widehat{R})$ with Affine Factors $A = 1.0$, $B = 2.0$

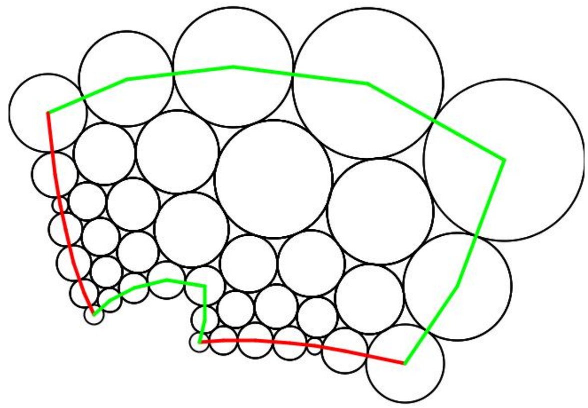

Figure 4.4: Circle Packing $P \leftrightarrow \widehat{K}(\widehat{R})$ with Affine Factors $A = 1.0$, $B = 4.0$

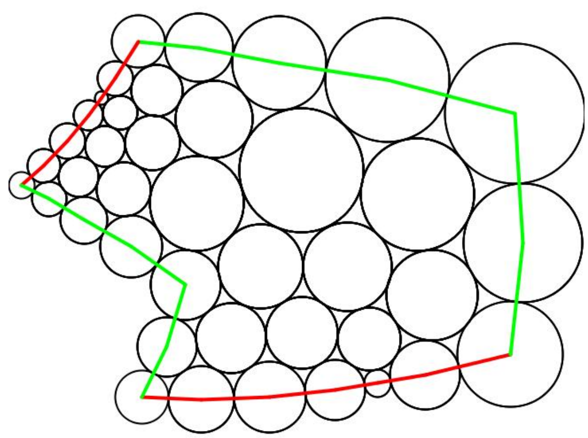

Figure 4.5: Circle Packing $P \leftrightarrow \widehat{K}(\widehat{R})$ with Affine Factors $A = 0.5$, $B = 2.0$

4.2 Circle Packings for Affine Tori

In this section it is shown that the circle packings of Section 4.1 may be interpreted as maximal circle packings on affine tori. A packing on a given surface is **maximal** if joining the centers of neighboring circles by geodesic segments results in a triangulation of the surface.

A conformal torus $T(\omega)$ has an associated intrinsic metric because its universal cover is the complex plane $\mathbb{C}$ and the group of covering transformations Λ is generated by isometries of the plane ($z \mapsto z+1$ and $z \mapsto z+\omega$). Hence it makes sense to consider circles and circle packings on $T(\omega)$. A circle packing for a combinatorial torus K on $T(\omega)$ may be specified by giving a circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$ such that centers of boundary circles satisfy:

$$z_i + 1 = z_i' \text{ for } i = 1, ..., n,$$
$$y_i + \omega = y_i' \text{ for } i = 1, ..., m,$$

where we have adopted the notational conventions from Section 4.1. Observe that the **side-pairings** for P are exactly the generators for Λ. Such a packing P projects locally isometrically from the covering space $\mathbb{C}$ to the torus $T(\omega)$ in such a way that each circle centered at z_i is identified with the circle centered at z_i' and similarly for each pair centered at y_i and y_i'.

The situation for affine tori is very different. Recall that an affine structure on a marked conformal torus $T(\omega, c)$ may be thought of as a **developing map** $f : \mathbb{C} \to \mathbb{C}$ where f is the identity mapping if $c = 0$ (in which case the affine torus is **flat**) and $f(z) = e^{cz}$ if $c \neq 0$. Since there is an entire family of distinct affine structures on a single conformal torus, and since a conformal torus does have an intrinsic metric, it follows that affine structures are not distinguished by an underlying metric. So there is no canonical metric on an affine torus. Indeed, when $c \neq 0$ the side-pairing maps of the developed image of a fundamental domain for $T(\omega, c)$ are not isometries of the plane, but merely automorphisms:

$$z \mapsto e^c z,$$
$$z \mapsto e^{c\omega} z.$$

Observe that these side-pairing maps are affine maps, and note that they are multiplication by a complex number only because of a normalization.

In the absence of a canonical metric, circles on affine tori cannot be defined in the usual manner. If $\pi : \mathbb{C} \to T(\omega, c)$ is the covering projection, then a circle on $T(\omega, c)$ is a homotopically trivial closed curve τ such that the developed image of each component of $\pi^{-1}(\tau)$ is a euclidean circle. Moreover, in the absence of a canonical metric, geodesics cannot be defined in the usual manner. But a curve τ on $T(\omega, c)$ may be considered a geodesic if the developed image of each component of $\pi^{-1}(\tau)$ is a geodesic.

It follows that a circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$ such that centers of boundary circles satisfy:

$$\alpha z_i + \gamma = z_i' \text{ for } i = 1, ..., n,$$
$$\beta y_i + \sigma = y_i' \text{ for } i = 1, ..., m,$$

may be interpreted as a circle packing of the developed image of a fundamental domain for an affine torus, and hence may also be interpreted as a circle packing on an affine torus. But by Theorem 4.1.9, the circle packings of Section 4.1 are exactly of this sort. Moreover, joining the centers of neighboring circles of $P \leftrightarrow \widehat{K}(\widehat{R})$ results in a triangulation of the developed image of a fundamental domain for $T(\omega, c)$, so the circle packing P may be interpreted as a maximal packing on the affine torus $T(\omega, c)$.

4.3 Side-Pairing Maps of Circle Packings for Affine Tori

In this section we explore some properties of side-pairing maps of circle packings on affine tori. We adopt the notational conventions from Section 4.1 and suppose that $P \leftrightarrow \widehat{K}(\widehat{R})$ is a circle packing constructed from an affine packing label $R \cdot S$ that is $\Gamma_1(A)$ and $\Gamma_2(B)$ as in Section 4.1.

Proposition 4.3.1. *The side-pairing maps $F(z) = \alpha z + \gamma$ and $G(z) = \beta z + \sigma$ for the circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$ satisfy $F \circ G = G \circ F$.*

Proof. Consider the packing $F(P)$. Since angles are preserved by F, it is clear that

$F(P) \leftrightarrow \widehat{K}(A \cdot \widehat{R})$. It follows that $F(P)$ has affine side-pairings F_F and G_F satisfying:

$$
\begin{aligned}
F_F(F(z_i)) &= F(z_i') \text{ for } i = 1, \ldots, n, \\
G_F(F(y_i)) &= F(y_i') \text{ for } i = 1, \ldots, m.
\end{aligned}
$$

These affine side-piring maps may be computed as in the proof of Theorem 4.1.9. Using the relations $F(z_i) = z_i'$ and $G(w_i) = w_i'$, the results of this calculation are:

$$
\begin{aligned}
F_F(z) &= F(z), \\
G_F(z) &= \beta z + \alpha\sigma + \gamma - \beta\gamma. \qquad (4.5)
\end{aligned}
$$

Similarly, the packing $G(P) \leftrightarrow \widehat{K}(B \cdot \widehat{R})$ has side-pairings F_G and G_G satisfying:

$$
\begin{aligned}
F_G(G(z_i)) &= G(z_i') \text{ for } i = 1, \ldots, n, \\
G_G(G(y_i)) &= G(y_i') \text{ for } i = 1, \ldots, m.
\end{aligned}
$$

Using the relations $F(z_i) = z_i'$ and $G(w_i) = w_i'$, these side-pairing maps may be computed as:

$$
\begin{aligned}
F_G(z) &= \alpha z + \beta\gamma + \sigma - \sigma\alpha, \qquad (4.6) \\
G_G(z) &= G(z).
\end{aligned}
$$

Let $t_1, \ldots, t_k$ be the centers of the circles of P for the vertices neighboring z_1 in $\widehat{K}$. Since $F_G \circ G(z_1) = G_F \circ F(z_1) = z_n'$, it is easy to see that the fourth condition of Proposition 4.1.5 implies that $F_G \circ G(t_i) = G_F \circ F(t_i)$ for $i = 1, \ldots, k$. It follows that $F_G \circ G = G_F \circ F$. But by (4.5) and (4.6) this is equivalent to:

$$\beta\gamma + \sigma = \alpha\sigma + \gamma.$$

This in turn implies that $F \circ G = G \circ F$. $\square$

Corollary 4.3.2. *The packing $F_G \circ G(P) = G_F \circ F(P)$ satisfies:*

$$F_G \circ G(P) \leftrightarrow \widehat{K}(AB \cdot \widehat{R}).$$

Figure 4.6 shows a circle packing P. Figure 4.7 shows the packing P and the packing $G(P)$. Figure 4.8 shows the packing P and the packing $F(P)$. And Figure 4.9 shows the packings P, $G(P)$, $F(P)$, and $F_G \circ G(P) = G_F \circ F(P)$.

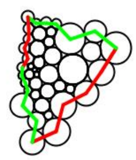

Figure 4.6: Circle Packing P

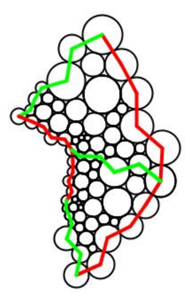

Figure 4.7: Circle Packings P and $G(P)$

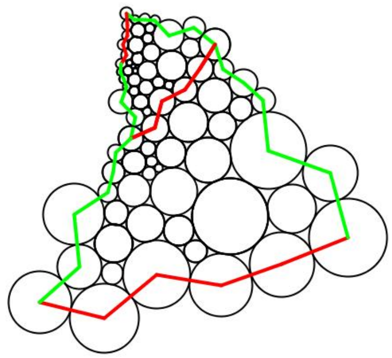

Figure 4.8: Circle Packings P and $F(P)$

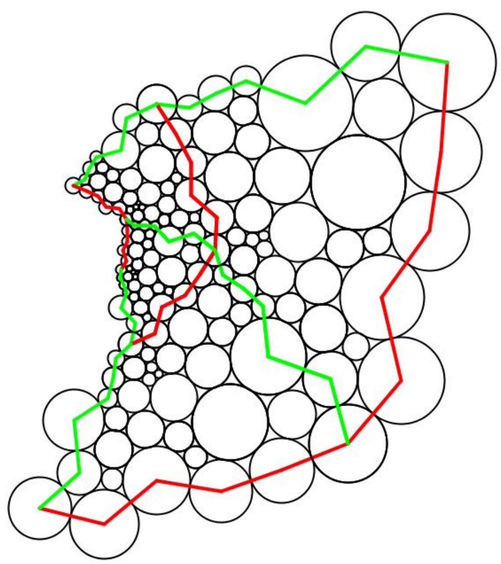

Figure 4.9: Circle Packings P, $F(P)$, $G(P)$, and $F_G \circ G(P) = G_F \circ F(P)$

Notation. An affine mapping of the complex plane takes the form $z \mapsto \alpha z + \gamma$, where $\alpha \neq 0$. Let $\mathcal{I}$ be the set containing just the identity mapping of the complex plane (so $\alpha = 1$ and $\gamma = 0$). Let $\mathcal{T}$ be the set of all such affine mappings that are non-identity translations. That is, $\alpha = 1$ and $\gamma \neq 0$. Let $\mathcal{A}$ be the set of all non-translation affine mappings. That is, $\alpha \neq 1$.

It is clear that these three sets partition the set of all affine mappings of the complex plane.

Proposition 4.3.3. *Suppose that $F(z) = \alpha z + \gamma$ and $G(z) = \beta z + \sigma$ are the side-pairing maps for the circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$. Then exactly one of the following possibilities occurs:*

1. *F and G are both in $\mathcal{A}$,*
2. *F and G are both in $\mathcal{T}$,*
3. *One of F and G is in $\mathcal{I}$ and the other is in $\mathcal{A}$.*

Proof. Observe that a scaling of the packing P has no effect on the classification of its side-pairings with respect to the partition under consideration. Observe also that by Proposition 4.1.7 the label $\widehat{R}$ is unique up to scaling. Let $A = |\alpha|$ and $B = |\beta|$. If $A = B = 1$, then by uniqueness the affine torus must be flat, and hence both its side-pairings must be in $\mathcal{T}$. It follows immediately that two cases cannot occur:

1. F and G both in $\mathcal{I}$,
2. One of F and G is in $\mathcal{I}$ and the other is in $\mathcal{T}$.

Suppose that $G \in \mathcal{A}$ and $F \notin \mathcal{A}$. By Proposition 4.3.1,

$$\beta\gamma + \sigma = \alpha\sigma + \gamma.$$

But $\alpha = 1$ since $F \notin \mathcal{A}$, so

$$\gamma(\beta - 1) = 0.$$

But $G \in \mathcal{A}$, so $\beta \neq 1$. It follows that $\gamma = 0$, and therefore $F \in \mathcal{I}$. It follows that another case is impossible: one of F and G is in $\mathcal{A}$ and the other is in $\mathcal{T}$.

It has been shown that three of the six cases are impossible. It remains to show that the other three cases do in fact occur. But this follows from our existence results and from the impossibility of the three cases just demonstated. Our existence results allow for the choice of positive affine factors A and B. If neither affine factor is 1, then F and G are both in $\mathcal{A}$. If $A = B = 1$, then F and G are both in $\mathcal{T}$. If exactly one of A or B is 1, then one of two cases occurs. Without loss of generality, suppose $A \neq 1$ and $B = 1$. If $\beta = 1$, then G is in $\mathcal{I}$ and the F is in $\mathcal{A}$. If $\beta \neq 1$, then F and G are both in $\mathcal{A}$ (and if the packing is normalized so that $\sigma = 0$, then G is a rotation). $\square$

Figure 4.2 is an example of the case F and G both in $\mathcal{T}$. Figure 4.5 is an example of the case F and G both in $\mathcal{A}$ and $\beta \neq 1$. Figure 4.4 is an example of the case F and G both in $\mathcal{A}$ and $\beta = 1$. It remains to give an example of the case one of F and G is in $\mathcal{I}$ and the other is in $\mathcal{A}$. In Figure 4.10 the affine factors are $A = 0.0024$ and $B = 1.0$. So the radii of the circles along the top green path are the same as the radii of the corresponding circles along the bottom green path ($B = 1.0$) and the radii of circles along the right red path are 0.0024 times the radii of the corresponding circles along the left red path ($A = 0.0024$). Note that the right red path and the circles along the right red path are not visible in this figure because they are so small relative to the circles along the left red path. In Figures 4.11 through 4.14 we zoom in so that the right red path is visible. Figure 4.11 is the result of zooming in on Figure 4.10. It is clear that F and G are both in $\mathcal{A}$ and that G is a rotation of less than 2π. Keeping B fixed at 1.0, we decrease A in increments of 0.0002. For the value $A = 0.0022$ it is still the case that F and G are both in $\mathcal{A}$ and that G is a rotation of less than 2π. For the value $A = 0.0020$ it is now the case that G is in $\mathcal{I}$ and the F is in $\mathcal{A}$ (G is a rotation of 2π). And for the value $A = 0.0018$ it becomes the case that F and G are both in $\mathcal{A}$ and that G is a rotation of more than 2π.

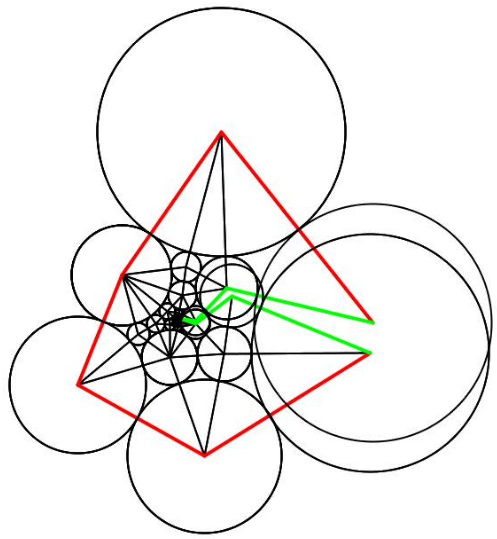

Figure 4.10: $A = 0.0024$, $B = 1.0$

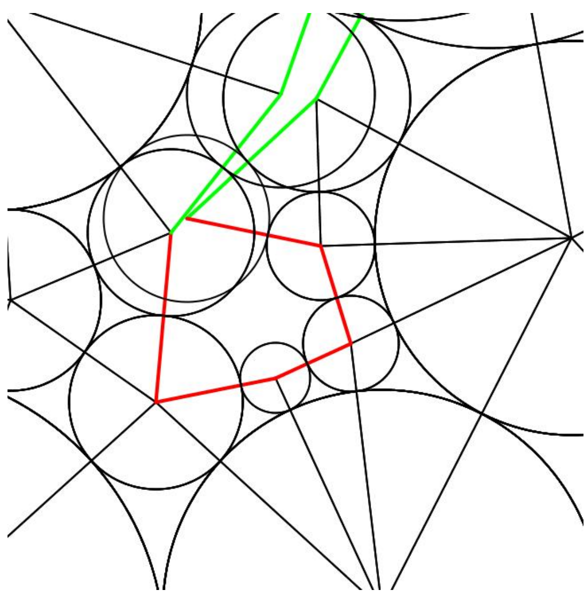

Figure 4.11: $A = 0.0024$, $B = 1.0$

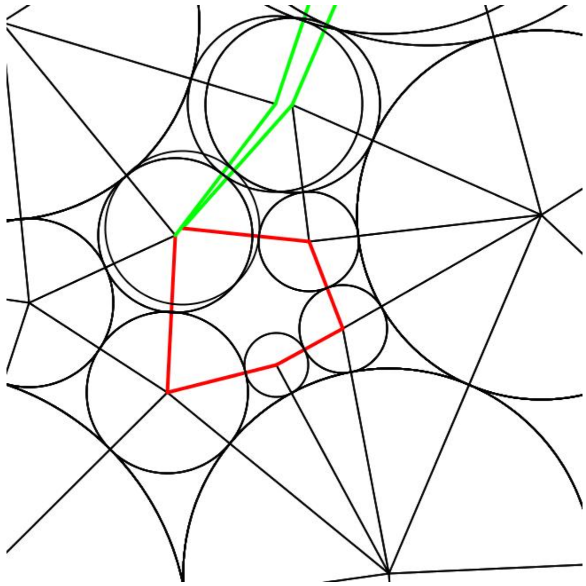

Figure 4.12: $A = 0.0022,\ B = 1.0$

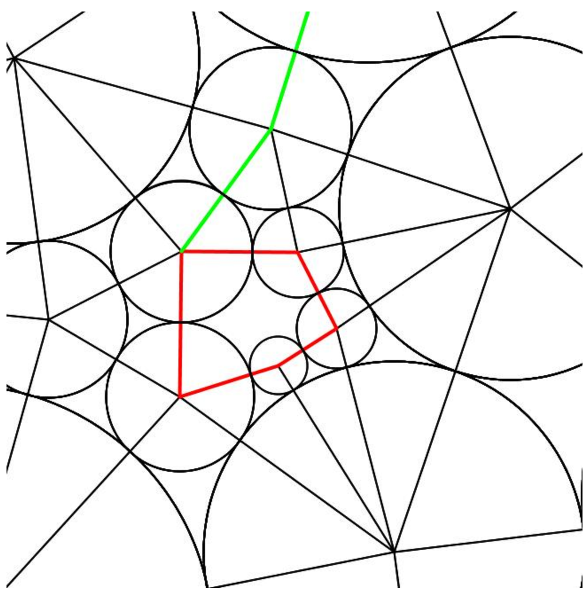

Figure 4.13: $A = 0.0020,\ B = 1.0$

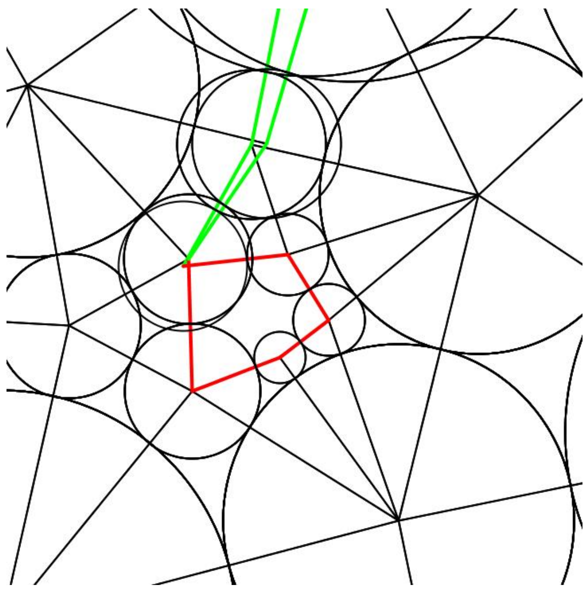

Figure 4.14: $A = 0.0018$, $B = 1.0$

4.4 Normalized Circle Packings for Affine Tori

In this section, a formula for moving a circle packing $P \leftrightarrow \widehat{K}(\widehat{R})$ for a combinatorial fundamental domain into a normalized position is given. It was shown in Section 4.2 that such a packing can be interpreted as a circle packing on an affine torus. In this section formulas for determining the Teichmüller and affine parameters of this torus from the normalized packing are given.

A fundamental domain for the torus $T(\omega)$ is the set

$$D = \{s + t\omega : 0 \leq s < 1 \text{ and } 0 \leq t < 1\}.$$

The corners for this fundamental domain are: 0, 1, ω, and $\omega + 1$.

Recall from Section 2.1 that the developing map for an affine torus satisfies:

$$f''/f' \equiv c$$

on $\mathbb{C}$. For $c \neq 0$ the most general solution to this differential equation is:

$$f(z) = k_1 e^{cz} + k_2,$$

where k_1 and k_2 are complex constants with $k_1 \neq 0$. The normalized developing map would have $k_1 = 1$ and $k_2 = 0$. Working with the non-normalized developing map, let the developed corners be $z_1 = f(0)$, $z_2 = f(1)$, $z_3 = f(\omega + 1)$, and $z_4 = f(\omega)$. A straightforward calculation shows that these developed corners may be moved into normalized position (that is, into the positions they would occupy if f had been normalized) by means of the normalizing map:

$$\Psi(z) = \frac{z_1 + z_3 - z_2 - z_4}{(z_1 - z_4)(z_1 - z_2)} z - \frac{z_1 z_3 - z_2 z_4}{(z_1 - z_4)(z_1 - z_2)}.$$

Consider the sides $\mu(s) = s$ and $\nu(t) = t\omega$ (for s and t in $[0, 1]$) of D. Suppose that their normalized developed images $f(\mu)$ and $f(\nu)$ are given. From this and the normalized positions of the developed corners, another straightforward calculation shows how to determine the affine parameter c and the Teichmüller parameter ω. Since the developed corners are in normalized position, $z_1 = 1$. Let

$$e^a = |z_2|,$$

$$b = \arg z_2,$$

where this argument is determined by taking $\arg z_1 = 0$ and letting the argument change continuously along $f(\mu)$. Similarly, let

$$e^M = |z_4|,$$
$$N = \arg z_4,$$

where the argument is determined by $f(\nu)$. If

$$x := \frac{aM + bN}{a^2 + b^2},$$

and

$$y := \frac{aN - bM}{a^2 + b^2},$$

then the affine and Teichmüller parameters are:

$$c = a + ib,$$
$$\omega = x + iy.$$

Returning now to the circle packing setting, the packing $P \leftrightarrow \widehat{K}(\widehat{R})$ for a combinatorial fundamental domain may be considered a packing of a developed image of a fundamental domain. Since Γ is known, the corners are known. So P can be moved into normalized position. Joining the centers of neighboring boundary circles of P gives piecewise linear paths between corners along which continuous changes in argument may be followed. So the affine and Teichmüller parameters may be computed from the packing in normalized position.

It is easy to check that in normalized position, the side-pairings for $P \leftrightarrow \widehat{K}(\widehat{R})$ are simply multiplication by a complex number (recall that $c \neq 0$, so there are no translations). A consequence is that (adopting the notation of Section 4.1):

$$z_1 = 1,$$
$$|z_1'| = A,$$
$$|z_n| = B,$$
$$|z_n'| = AB.$$

See Figure 4.15 for an example of a packing in normalized position. In this figure, $A = 0.3$ and $B = 1.0$, so as expected $z_1 = 1$, z_n lies on the unit circle (shown in the figure), and z_1' and z_n' lie on the circle of modulus 0.3.

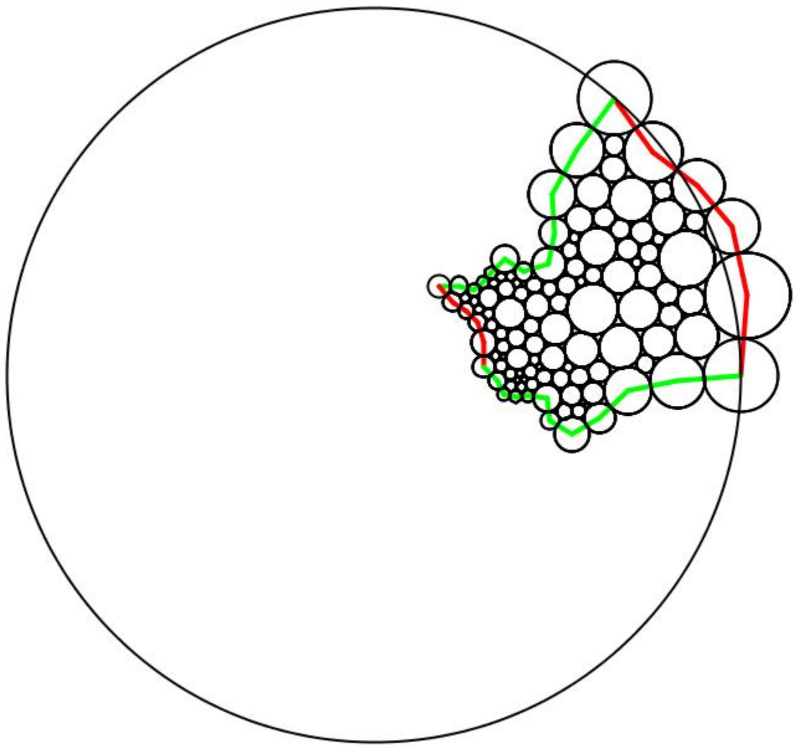

Figure 4.15: Normalized Packing with Affine Factors $A = 0.3$, $B = 1.0$

Chapter 5

Continuity Results

5.1 Two Error Measurement Definitions for Vertex Labels

Definition 5.1.1. Let R be a vertex label for a combinatorial torus K, and let $v \in K$. The **signed angle error at** v is

$$E_R(v) := \theta_R(v) - 2\pi.$$

Definition 5.1.2. Let R be a vertex label for a combinatorial torus K. The **angle error of** R is

$$E(R) := \sum_{v \in K} |E_R(v)|.$$

Lemma 5.1.3. *If K is a combinatorial torus, then*

$$\sum_{v \in K} E_R(v) = 0.$$

Proof. Since K is a combinatorial torus, by counting $F = 2V$. Therefore,

$$\begin{aligned}
2\pi V &= \pi F, \\
&= \sum_{v \in K} \theta_R(v), \\
&= \sum_{v \in K} \left[2\pi + E_R(v)\right], \\
&= \sum_{v \in K} 2\pi + \sum_{v \in K} E_R(v), \\
&= 2\pi V + \sum_{v \in K} E_R(v).
\end{aligned}$$

The conclusion follows. $\square$

Definition 5.1.4. Let R be a vertex label for a combinatorial torus K. Fix $v_0 \in K$ and let $\widetilde{R}$ be the vertex packing label for K such that $\widetilde{R}(v_0) = R(v_0)$. Define

$$M(R) := \max_{v \in K} \left\{ \max \left\{ \frac{\widetilde{R}(v)}{R(v)}, \frac{R(v)}{\widetilde{R}(v)} \right\} \right\}.$$

Definition 5.1.5. Let R, v_0, and $\widetilde{R}$ be as in Definition 5.1.4. Define the **deviation of** R **from** $\widetilde{R}$ by:

$$\|R - \widetilde{R}\| := M(R) - 1.$$

The following lemma is clear from the previous definitions.

Lemma 5.1.6. *If R, v_0, and $\widetilde{R}$ are as in Definition 5.1.4, then $M(R) \geq 1$ and $\|R - \widetilde{R}\| \geq 0$.*

5.2 A Continuity Result for Vertex Label Error Measurements

For this section, let K be a fixed combinatorial torus and v_0 a fixed vertex of K. All labels R considered in this section will be vertex labels for K and will have $R(v_0) = 1$. And $\widetilde{R}$ will be the vertex packing label for K such that $\widetilde{R}(v_0) = 1$.

Proposition 5.2.1. *Let $v \in K\backslash\{v_0\}$ and $r > 0$. Then there exists a unique vertex label S such that:*

$$\begin{aligned} S(v_0) &= 1, \\ S(v) &= r, \\ E_S(w) &= 0 \textit{ for all } w \in K\backslash\{v_0, v\}. \end{aligned}$$

Proof. Let $K^* := K\backslash\{v_0, v\}$. The existence argument is just a modification of the proof of the existence of a euclidean packing label for K in Section 3.1. In that proof we fixed the radius of v_0 and modified the other radii to force their angle sums to 2π. Now we fix the radii of v_0 and v and modify the other radii to force their angle sums to 2π. Since K^* is connected, the modified proof is valid.

To show uniqueness, suppose that $S1$ and $S2$ are labels satisfying the desired properties. Suppose that $S1 \neq S2$. There are two cases.

Case I: one label is strictly larger than the other at every vertex in K^*. Without loss of generality, suppose $S1(w) < S2(w)$ for all $w \in K^*$. By monotonicity, then, $E_{S1}(v_0) < E_{S2}(v_0)$ and $E_{S1}(v) < E_{S2}(v)$. So

$$E_{S1}(v_0) + E_{S1}(v) < E_{S2}(v_0) + E_{S2}(v). \tag{5.1}$$

But $E_{S1}(w) = 0$ for all $w \in K^*$, so by Lemma 5.1.3,

$$E_{S1}(v_0) + E_{S1}(v) = \sum_{w \in K} E_{S1}(w) = 0. \tag{5.2}$$

Similarly,

$$E_{S2}(v_0) + E_{S2}(v) = \sum_{w \in K} E_{S2}(w) = 0. \tag{5.3}$$

The statements (5.1), (5.2), and (5.3) cannot all be true. So Case I leads to a contradiction.

Case II: if Case I does not hold, we may suppose without loss of generality that there exist $w, w' \in K^*$ such that $S1(w) < S2(w)$ and $S2(w') \leq S1(w')$. And since K^* is connected, we may suppose that w and w' are neighbors. Let m be the label defined by $m(u) = \min\{S1(u), S2(u)\}$ for all $u \in K$. By monotonicity of angle sums and our assumptions on $S1$ and $S2$,

$$\theta_m(u) \leq 2\pi \text{ for all } u \in K^*. \tag{5.4}$$

Moreover, since $m(v_0) = S1(v_0) = 1$ and $m(v) = S1(v) = r$, by monotonicity of angle sums and assumptions on $S1$ it follows that:

$$\theta_m(v_0) + \theta_m(v) \leq \theta_{S1}(v_0) + \theta_{S1}(v) = 2\pi + E_{S1}(v_0) + 2\pi + E_{S1}(v) = 4\pi. \tag{5.5}$$

But by monotonicity of angle sums and our assumptions on w and w', $\theta_m(w') < \theta_{S2}(w') = 2\pi$. So by (5.4) and (5.5),

$$\begin{aligned}\sum_{u \in K} \theta_m(u) &= \theta_m(v_0) + \theta_m(v) + \sum_{u \in K^*} \theta_m(u), \\ &\leq 4\pi + \sum_{u \in K^*} \theta_m(u), \\ &\leq 2\pi(V-1) + \theta_m(w'), \\ &< 2\pi(V-1) + 2\pi.\end{aligned}$$

This is a contradiction, since the total angle sum of m must be $2\pi V$.

Since both cases led to a contradiction, we conclude that $S1 = S2$. □

In view of Proposition 5.2.1, the following definition is coherent.

Definition 5.2.2. Let $v \in K \backslash \{v_0\}$ and $r > 0$. Then $S[v,r]$ is the vertex label such that:

$$\begin{aligned}&S[v,r](v_0) = 1, \\ &S[v,r](v) = r, \\ &E_{S[v,r]}(w) = 0 \text{ for all } w \in K \backslash \{v_0, v\}.\end{aligned}$$

Lemma 5.2.3. *Let $u \in K \backslash \{v_0\}$ and suppose $R1$ and $R2$ are labels such that:*

$$\begin{gathered}R1(w) = R2(w) \text{ for all } w \in K \backslash \{u\}, \\ \theta_{R1}(u) < \theta_{R2}(u) < 2\pi.\end{gathered}$$

Then $E(R2) \leq E(R1)$.

Proof. Let the neighboring vertices of u be $u_1, ..., u_m$. If a vertex w is not u or a neighbor of u, then clearly

$$E_{R1}(w) = E_{R2}(w). \tag{5.6}$$

By Lemma 5.1.3, then,

$$\sum_{i=1}^{m}(E_{R1}(u_i) - E_{R2}(u_i)) = E_{R2}(u) - E_{R1}(u). \tag{5.7}$$

By definition of signed angle error, $E_{R1}(u) < E_{R2}(u) < 0$. By monotonicity of angle sums, $R2(u) < R1(u)$, so $E_{R2}(u_i) < E_{R1}(u_i)$ for $i = 1, ..., m$. Hence the number (5.7) is positive. It follows that:

$$\begin{aligned}
|E_{R2}(u)| + \sum_{i=1}^{m}|E_{R2}(u_i)| &= |E_{R2}(u)| + \sum_{i=1}^{m}|E_{R2}(u_i) + E_{R1}(u_i) - E_{R1}(u_i)|, \\
&\leq |E_{R2}(u)| + \sum_{i=1}^{m}|E_{R1}(u_i) - E_{R2}(u_i)| + \sum_{i=1}^{m}|E_{R1}(u_i)|, \\
&= -E_{R2}(u) + \sum_{i=1}^{m}(E_{R1}(u_i) - E_{R2}(u_i)) + \sum_{i=1}^{m}|E_{R1}(u_i)|, \\
&= -E_{R2}(u) + E_{R2}(u) - E_{R1}(u) + \sum_{i=1}^{m}|E_{R1}(u_i)|, \\
&= -E_{R1}(u) + \sum_{i=1}^{m}|E_{R1}(u_i)|, \\
&= |E_{R1}(u)| + \sum_{i=1}^{m}|E_{R1}(u_i)|.
\end{aligned}$$

The conclusion follws from this and (5.6). □

The proofs of the following three lemmas are similar to the proof of Lemmma 5.2.3.

Lemma 5.2.4. *Let $u_i \in K\backslash\{v_0\}$ for $i = 1, ..., n$ and suppose $R1$ and $R2$ are labels such that:*

$$R1(w) = R2(w) \text{ for all } w \in K\backslash\{u_1, ..., u_n\},$$
$$R2(u_i) < R1(u_i) \text{ for } i = 1, ..., n,$$
$$\theta_{R1}(u_i) < \theta_{R2}(u_i) < 2\pi \text{ for } i = 1, ..., n.$$

Then $E(R2) \leq E(R1)$.

Lemma 5.2.5. *Let $u \in K\backslash\{v_0\}$ and suppose $R1$ and $R2$ are labels such that:*

$$R1(w) = R2(w) \text{ for all } w \in K\backslash\{u\},$$
$$2\pi < \theta_{R2}(u) < \theta_{R1}(u).$$

Then $E(R2) \leq E(R1)$.

Lemma 5.2.6. *Let $u_i \in K\backslash\{v_0\}$ for $i = 1, ..., n$ and suppose $R1$ and $R2$ are labels such that:*

$$R1(w) = R2(w) \text{ for all } w \in K\backslash\{u_1, ..., u_n\},$$
$$R1(u_i) < R2(u_i) \text{ for } i = 1, ..., n,$$
$$2\pi < \theta_{R2}(u) < \theta_{R1}(u) \text{ for } i = 1, ..., n.$$

Then $E(R2) \leq E(R1)$.

Definition 5.2.7. If $L \subset K$ and if $w \in K\backslash L$, then w **is a neighbor of** L, written '$w \sim L$', means that there exists $u \in L$ such that $\langle u, w\rangle$ is an edge of K.

Proposition 5.2.8. *If R is a vertex label then for any $v \in K\backslash\{v_0\}$,*

$$E(S[v, R(v)]) \leq E(R).$$

Proof. Let $K^* := K\backslash\{v_0, v\}$. For notational convenience, let $R0$ be the given starting vertex label. Our strategy will be to modify the label $R0$ until reaching $S[v, R(v)]$. Each modification will be non-increasing in label angle error. We will alternate between finite process modifications and infinite process modifications.

We begin with a finite process modification. Define the following sets:

$$\begin{aligned}
\mathcal{S}^1 &= \{w \in K^* : \theta_{R0}(w) \leq 2\pi\},\\
\mathcal{T}^1 &= \{w \in \mathcal{S}^1 : \theta_{R0}(w) < 2\pi\},\\
\mathcal{U}^1 &= \{w \in \mathcal{S}^1 : w \in \mathcal{T}^1 \text{ or there is an edge path } \{w_1, ..., w_n\}\\
&\qquad\qquad \text{such that } w_1 = w,\ w_n \in \mathcal{T}^1, \text{ and } w_i \in \mathcal{S}^1 \text{ for } i = 1, ..., n\}.
\end{aligned}$$

Observe that

$$\mathcal{T}^1 \subset \mathcal{U}^1 \subset \mathcal{S}^1 \subset K^*.$$

Lemma 5.2.9. *If $w \in \mathcal{S}^1\backslash\mathcal{U}^1$ then no neighbor of w is an element of $\mathcal{U}^1$.*

Proof. Suppose w has a neighbor $w' \in \mathcal{U}^1$. Then either $w' \in \mathcal{T}^1$ or there is an edge path connecting w' to $\mathcal{T}^1$ through $\mathcal{S}^1$. In either case it follows that there is an edge path connecting w to $\mathcal{T}^1$ through $\mathcal{S}^1$ and hence that $w \in \mathcal{U}^1$. Contradiction. $\square$

Put $\mathcal{T}^11 = \mathcal{T}^1$ and suppose $\mathcal{T}^11$ is nonempty. Fix $w \in \mathcal{T}^11$ for the moment. Modify the label $R0$ at w so that the angle sum at w increases but remains less than 2π. By Lemma 5.2.3, this modification is non-increasing in label angle error. Observe that this modification results in a decrease in the radius of w and hence results in an angle sum decrease at the neighbors of w, so all angle sums that were less than 2π before the modification remain so after the modification.

So we may repeat this process, visiting each vertex of $\mathcal{T}^11$ exactly once. Let the resulting label be $R01$. By construction, $R0 = R01$ on $K \backslash \mathcal{U}^1$. So if $w \in \mathcal{S}^1 \backslash \mathcal{U}^1$, it follows by Lemma 5.2.9 that w has the same angle sum (namely 2π) with respect to $R0$ and $R01$.

By construction, the label $R01$ has the following properties:

$$
\begin{aligned}
&R01(w) < R0(w) \text{ for all } w \in \mathcal{T}^11,\\
&R01(w) = R0(w) \text{ for all } w \in K\backslash\mathcal{T}^11,\\
&\theta_{R01}(w) < 2\pi \text{ for all } w \in \mathcal{T}^11,\\
&\theta_{R01}(w) < \theta_{R0}(w) \text{ for any } w \in K\backslash\mathcal{T}^11 \text{ such that } w \sim \mathcal{T}^11,\\
&\theta_{R01}(w) < 2\pi \text{ for any } w \in \mathcal{S}^1\backslash\mathcal{T}^11 \text{ such that } w \sim \mathcal{T}^11,\\
&\theta_{R01}(w) = 2\pi \text{ for all } w \in \mathcal{S}^1\backslash\mathcal{U}^1,\\
&E(R01) \leq E(R0).
\end{aligned}
\tag{5.8}
$$

Observe that any vertex w satisfying the conditions of (5.8) is necessarily an element of $\mathcal{U}^1$.

Now define the set

$$\mathcal{T}^12 = \{w \in \mathcal{S}^1 : \theta_{R01}(w) < 2\pi\}.$$

Suppose $\mathcal{T}^12\backslash\mathcal{T}^11$ is nonempty and consider a vertex $w \in \mathcal{T}^12\backslash\mathcal{T}^11$. Since $w \in \mathcal{T}^12$, $w \in \mathcal{S}^1$. So $\theta_{R0}(w) \leq 2\pi$. But $w \notin \mathcal{T}^11$, so we must have $\theta_{R0}(w) = 2\pi$. Now $w \notin \mathcal{T}^11$ also means that $R01(w) = R0(w)$, so in order for $\theta_{R01}(w) < 2\pi$ to hold, it must be the case that $R01(w') < R0(w')$ for some neighbor w' of w. By construction of $R01$, it follows that $w' \in \mathcal{T}^11$. So w satisfies the conditions of (5.8) and is therefore in $\mathcal{U}^1$.

It follows that:

$$\mathcal{T}^1 1 \subset \mathcal{T}^1 2 \subset \mathcal{U}^1.$$

In fact, if w satisfies the conditions of (5.8), it is by definition an element of $\mathcal{T}^1 2 \backslash \mathcal{T}^1 1$, so that:

$$\mathcal{T}^1 2 \backslash \mathcal{T}^1 1 = \{w \in \mathcal{S}^1 : w \notin \mathcal{T}^1 1 \text{ and } w \sim \mathcal{T}^1 1\}.$$

Now visit each vertex of $\mathcal{T}^1 2$ exactly once, making adjustments to obtain a label $R02$ such that:

$$\begin{aligned}
&R02(w) < R01(w) \text{ for all } w \in \mathcal{T}^1 2,\\
&R02(w) = R01(w) \text{ for all } w \in K \backslash \mathcal{T}^1 2,\\
&\theta_{R02}(w) < 2\pi \text{ for all } w \in \mathcal{T}^1 2,\\
&\theta_{R02}(w) < \theta_{R01}(w) \text{ for any } w \in K \backslash \mathcal{T}^1 2 \text{ such that } w \sim \mathcal{T}^1 2,\\
&\theta_{R02}(w) < 2\pi \text{ for any } w \in \mathcal{S}^1 \backslash \mathcal{T}^1 2 \text{ such that } w \sim \mathcal{T}^1 2,\\
&\theta_{R02}(w) = 2\pi \text{ for all } w \in \mathcal{S}^1 \backslash \mathcal{U}^1,\\
&E(R02) \leq E(R01) \leq E(R0).
\end{aligned}$$

Repeating this process we obtain sets

$$\mathcal{T}^1 n \subset \mathcal{U}^1 \text{ for } n = 1, 2, ...,$$

where in general $\mathcal{T}^1[n+1]$ is formed by adding to $\mathcal{T}^1 n$ any vertices of $\mathcal{S}^1$ that are not in $\mathcal{T}^1 n$ but are neighbors of $\mathcal{T}^1 n$. For any $u \in \mathcal{U}^1$, by definition of $\mathcal{U}^1$ and construction of the sets $\mathcal{T}^1 n$, there is an $M(u) > 0$ such that $u \in \mathcal{T}^1[M(u)]$. Let

$$N := \max_{u \in \mathcal{U}^1} \{M(u)\}.$$

Since $\mathcal{T}^1 n \subset \mathcal{T}^1[n+1]$ for $n = 1, 2, ...$, it follows that $\mathcal{T}^1 N = \mathcal{U}^1$. Put $R1 := R0N$ and observe that:

$$\begin{aligned}
&\mathcal{T}^1 N = \mathcal{U}^1,\\
&\theta_{R1}(w) < 2\pi \text{ for all } w \in \mathcal{U}^1, && (5.9)\\
&\theta_{R1}(w) = 2\pi \text{ for all } w \in \mathcal{S}^1 \backslash \mathcal{U}^1, && (5.10)\\
&E(R1) \leq E(R0). && (5.11)
\end{aligned}$$

This completes the finite process modification.

We now perform an infinite process modification. Define the collection of labels

$$\Phi 2 = \{R : \theta_R(w) < 2\pi \text{ for } w \in \mathcal{U}^1,\ R(w) = R1(w) \text{ for } w \in K \backslash \mathcal{U}^1\}.$$

By (5.9), $R1 \in \Phi 2$. So we may define the label $R2$ by:

$$R2(w) = \inf_{R \in \Phi 2} R(w)$$

for all $w \in K$.

Lemma 5.2.10. $R2(w) > 0$ *for all* $w \in K$.

Proof. Similar to arguments in Section 3.1 since $K \backslash \mathcal{U}^1$ is nonempty. □

Lemma 5.2.11. $\theta_R(w) = 2\pi$ *for any* $R \in \Phi 2$ *and for any* $w \in \mathcal{S}^1 \backslash \mathcal{U}^1$.

Proof. This follows from definition of $\Phi 2$, (5.10), and Lemma 5.2.9. □

Lemma 5.2.12. $\theta_{R2}(w) = 2\pi$ *for all* $w \in \mathcal{S}^1$.

Proof. If $w \in \mathcal{U}^1$, this follows from the definition of $\Phi 2$, monotonicity of angle sums, and continuity of angle sums. If $w \in \mathcal{S}^1 \backslash \mathcal{U}^1$, this follows from Lemma 5.2.11. □

By monotonicity of angle sums, $\Phi 2$ is closed under min, so there is a sequence of labels R_n such that $R_n \in \Phi 2$ and $R_n(w) \to R2(w)$ as $n \to \infty$ for all $w \in K$. Since $R1 \in \Phi 2$, we may suppose $R_1 = R1$. By continuity of angle sums,

$$\theta_{R_n}(w) \to 2\pi \text{ as } n \to \infty \text{ for all } w \in \mathcal{U}^1. \tag{5.12}$$

Since each $R_n \in \Phi 2$, we may assume (taking subsequences if necessary) that the sequences in (5.12) are strictly increasing. So

$$\theta_{R_1}(w) < \theta_{R_2}(w) < ... < 2\pi \text{ for all } w \in \mathcal{U}^1, \tag{5.13}$$

$$R_1(w) = R_2(w) = ... \text{ for all } w \in K \backslash \mathcal{U}^1. \tag{5.14}$$

We may of course assume that $R_n(w)$ is decreasing for all $w \in \mathcal{U}^1$, so from (5.13) and (5.14) it follows by Lemma 5.2.4 that:

$$E(R_{n+1}) \leq E(R_n) \text{ for } n = 1, 2, ... \tag{5.15}$$

But $R1 = R_1$ and $R_n \to R2$ as $n \to \infty$, so by continuity of angle sums and (5.15), $E(R2) \le E(R1)$. In summary, the label $R2$ satisfies:

$$\theta_{R2}(w) = 2\pi \text{ for all } w \in \mathcal{S}^1, \tag{5.16}$$

$$E(R2) \le E(R1). \tag{5.17}$$

This completes the infinite process modification.

Remark. We assumed above that $\mathcal{T}^1$ was nonempty. If $\mathcal{T}^1 = \emptyset$ then $\mathcal{U}^1 = \emptyset$, and the above construction trivially yields $R0 = R1 = R2$. In this case the properties (5.10), (5.11), (5.16), and (5.17) are trivially satisfied.

Now iterate pairs of finite and infinite process modifications, obtaining labels Rn and sets

$$\mathcal{S}^{2n+1} = \{w \in K^* : \theta_{R[2n]}(w) \le 2\pi\}$$

for $n = 0, 1, 2, ...$ such that:

$$\begin{aligned}
&\theta_{R[2n+2]}(w) = 2\pi \text{ for all } w \in \mathcal{S}^{2n+1},\\
&E(R[n+1]) \le E(Rn),\\
&\mathcal{S}^{2n+1} \subset \mathcal{S}^{2n+3} \subset K^*.
\end{aligned}$$

Since K is finite, after finitely many iterations, we reach a stage at which $\mathcal{S}^{2n+1} = \mathcal{S}^{2n+3}$. It is easy to check that at this stage all angle sums of K^* vertices are $\ge 2\pi$ with respect to the label $R[2n+2]$. If any K^* vertices have angle sums $> 2\pi$ with respect to $R[2n+2]$, a finite process modification will produce a label S such that all K^* vertices have angle sums $> 2\pi$ with respect to S and (by Lemma 5.2.5) such that $E(S) \le E(R[2n+2])$. Finally, an infinite process modification will produce the label $S[v, R(v)]$ and guarantee (by Lemma 5.2.6) that:

$$E(S[v, R(v)]) \le E(S) \le E(R[2n+2]) \le E(R0).$$

So $E(S[v, R(v)]) \le E(R)$. This completes the proof of the Proposition. $\square$

Definition 5.2.13. For $M \ge 1$ define

$$m(M) := \min_{w \in K \setminus \{v_0\}} \left\{ \min \left\{ E[S(w, M \cdot \widetilde{R}(w))], E\left[S\left(w, \frac{1}{M} \cdot \widetilde{R}(w) \right) \right] \right\} \right\}.$$

Lemma 5.2.14. *If $M > 1$, then $m(M) > 0$.*

Proof. This follows from the definition of $m(M)$ and the uniqueness result of Section 3.5. $\square$

Proposition 5.2.15. *For any vertex label R, $m(M(R)) \leq E(R)$.*

Proof. By definition of $M(R)$, there is a vertex $w_1 \in K \backslash \{v_0\}$ such that either

$$\widetilde{R}(w_1) = M(R) \cdot R(w_1), \tag{5.18}$$

or

$$R(w_1) = M(R) \cdot \widetilde{R}(w_1).$$

Suppose (5.18) is true. By definition of $m(M(R))$, there is a vertex $w_2 \in K \backslash \{v_0\}$ such that either

$$m(M(R)) = E[S(w_2, M(R) \cdot \widetilde{R}(w_2))], \tag{5.19}$$

or

$$m(M(R)) = E\left[S\left(w_2, \frac{1}{M(R)} \cdot \widetilde{R}(w_2)\right)\right].$$

Suppose (5.19) is true. Then by definition of $m(M(R))$, (5.18), and Proposition 5.2.8,

$$\begin{aligned} m(M(R)) &= E[S(w_2, M(R) \cdot \widetilde{R}(w_2))], \\ &\leq E\left[S\left(w_1, \frac{1}{M(R)} \cdot \widetilde{R}(w_1)\right)\right], \\ &= E[S(w_1, R(w_1))], \\ &\leq E(R). \end{aligned}$$

The other cases are similar. $\square$

Proposition 5.2.16. *If $1 \leq M_1 \leq M_2$, then $m(M_1) \leq m(M_2)$.*

Proof. Fix $v \in K \backslash \{v_0\}$ and let $K^* = K \backslash \{v_0, v\}$. We need two lemmas.

Lemma 5.2.17. *If $\widetilde{R}(v) < r < r'$, then $E[S(v, r)] < E[S(v, r')]$.*

Proof. Let T be the label obtained from $\widetilde{R}$ by increasing the radius of v to r and leaving all other radii fixed. The neighbors of v in K^* all have angle sums $> 2\pi$ with respect to T. Non-neighboring vertices of v in K^* all have angle sums 2π with respect to T. It follows by monotonicity of angle sums, connectedness of K^*, and Proposition 5.2.1 that:

$$\widetilde{R}(w) = T(w) < S(v, r)(w) \text{ for all } w \in K^*.$$

From this it follows that:

$$2\pi < \theta_{S(v,r)}(v_0). \tag{5.20}$$

Let T' be the label obtained from $S(v,r)$ by increasing the radius of v to r' and leaving all other radii fixed. The neighbors of v in K^* all have angle sums $> 2\pi$ with respect to T'. Non-neighboring vertices of v in K^* all have angle sums 2π with respect to T'. It follows by monotonicity, connectedness of K^*, and the uniqueness statement of Proposition 5.2.1 that:

$$S(v,r)(w) = T'(w) < S(v,r')(w) \text{ for all } w \in K^*.$$

From this it follows that:

$$\theta_{S(v,r)}(v_0) < \theta_{S(v,r')}(v_0). \tag{5.21}$$

By (5.20) and (5.21),

$$0 < E_{S(v,r)}(v_0) < E_{S(v,r')}(v_0). \tag{5.22}$$

Since the signed angle error of $S(v,r)$ at v_0 is positive, by Lemma 5.1.3 we have:

$$E[S(v,r)] = |E_{S(v,r)}(v_0)| + |E_{S(v,r)}(v)| = E_{S(v,r)}(v_0) - E_{S(v,r)}(v) = 2E_{S(v,r)}(v_0).$$

Similarly,

$$E[S(v,r')] = |E_{S(v,r')}(v_0)| + |E_{S(v,r')}(v)| = E_{S(v,r')}(v_0) - E_{S(v,r')}(v) = 2E_{S(v,r')}(v_0).$$

So the conclusion follows from (5.22). This completes the proof of the lemma. □

Lemma 5.2.18. *If $r' < r < \widetilde{R}(v)$, then $E[S(v,r)] < E[S(v,r')]$.*

Proof. Similar to proof of Lemma 5.2.17. □

By Lemma 5.2.17,

$$E[S(w, M_1 \cdot \widetilde{R}(w))] \leq E[S(w, M_2 \cdot \widetilde{R}(w))]. \tag{5.23}$$

And by Lemma 5.2.18,

$$E\left[S\left(w, \frac{1}{M_1} \cdot \widetilde{R}(w)\right)\right] \leq E\left[S\left(w, \frac{1}{M_2} \cdot \widetilde{R}(w)\right)\right]. \tag{5.24}$$

Since (5.23) and (5.24) hold for any $v \in K \backslash \{v_0\}$, it follows that $m(M_1) \leq m(M_2)$. This completes the proof of the proposition. $\square$

The main result of this section shows that the deviation of R from $\widetilde{R}$ can be forced to be arbitrarily small by making the angle error of R sufficiently small.

Theorem 5.2.19. *For each $\epsilon > 0$ there exists a $\delta > 0$ such that:*

$$E(R) < \delta \Rightarrow \|R - \widetilde{R}\| < \epsilon.$$

Proof. Let $\epsilon > 0$. Since $\epsilon + 1 > 1$, by Lemma 5.2.14 $m(\epsilon + 1) > 0$. Suppose now that:

$$E(R) < m(\epsilon + 1). \tag{5.25}$$

Suppose towards a contradiction that $\|R - \widetilde{R}\| \geq \epsilon$. Since $\|R - \widetilde{R}\| = M(R) - 1$, it follows that:

$$M(R) \geq \epsilon + 1 > 1.$$

So by Proposition 5.2.16,

$$m(M(R)) \geq m(\epsilon + 1). \tag{5.26}$$

And by Proposition 5.2.15,

$$m(M(R)) \leq E(R). \tag{5.27}$$

But (5.25) and (5.27) imply that $m(M(R)) < m(\epsilon + 1)$, contradicting (5.26). It follows that:

$$E(R) < m(\epsilon + 1) \Rightarrow \|R - \widetilde{R}\| < \epsilon.$$

So choosing $\delta = m(\epsilon + 1)$ completes the argument. $\square$

5.3 Side-Pairing Parameters are Continuous in Side-Pairing Moduli

Throughout this section fix K a combinatorial torus, $v_0 \in K$, and $\Gamma = \Gamma_1 * \Gamma_2$ a fundamental pair. As in Section 3.3, for given affine factors A and B let R be the vertex label and S the face label such that $R \cdot S$ is an affine packing label that is $\Gamma_1(A)$ and $\Gamma_2(B)$. All vertex labels in this section will assign radius 1 to v_0.

Similarly, for given positive A' and B', R' will be the corresponding vertex label and S' the corresponding face label.

Lemma 5.3.1. *Let A and B be positive numbers. There exists a neigborhood of (A, B) and there exist positive numbers l and L such that for any (A', B') in the neighborhood, $l \leq R'(v) \leq L$ for all $v \in K$.*

Proof. Suppose that the statement concerning the upper bound L is false. Since K is finite, this implies that there exists a vertex $w \in K \backslash \{v_0\}$, sequences of affine parameters A_n and B_n, and corresponding sequences of vertex labels R_n and face labels S^n such that:

$$A_n \to A \text{ as } n \to \infty, \tag{5.28}$$

and

$$B_n \to B \text{ as } n \to \infty, \tag{5.29}$$

and

$$R_n(w) \to \infty \text{ as } n \to \infty. \tag{5.30}$$

By our construction of the face labels S^n and S in Section 3.3, it follows from (5.28) and (5.29) that:

$$S^n_f(v) \to S_f(v) \text{ as } n \to \infty \tag{5.31}$$

for all $v \in K$ and all $f \in F(v)$. By construction, all entries of S are finite and positive, so (5.30) and (5.31) imply that:

$$(R_n \cdot S^n)_f(w) \to \infty \text{ as } n \to \infty \tag{5.32}$$

for all $f \in F(w)$. But $R_n(v_0) = 1$ for $n = 1, 2, ...$ and hence

$$(R_n \cdot S^n)_f(v_0) \to S(v_0) < \infty \text{ as } n \to \infty \tag{5.33}$$

for all $f \in F(v_0)$. Again by our construction in Section 3.3,

$$\theta_{(R_n \cdot S^n)}(v) = 2\pi \text{ for all } v \in K. \tag{5.34}$$

Taking subsequences, we may suppose that for all $v \in K$ the sequence $R_n(v)$ converges to a number in $[0, \infty]$. It then follows from (5.32) and (5.33), arguing as in the proof of Proposition 3.1.9, that we get a contradiction to (5.34). We conclude that the statement concerning the upper bound L is true.

Similarly, if the statement concerning the lower bound l is false, we get a contradiction to (5.34) by arguing as in the proof of Proposition 3.1.2. We conclude that the statement about the lower bound l is true. $\square$

Remark. It is clear that the definitions of signed angle error and (total) angle error given in Section 5.1 can be understood as applying to face labels (just interpret the angle sums in the definitions as relative to face labels instead of vertex labels). These definitions will be used in this generalized sense below.

Notation. Let $\mathbb{R}_+$ be the set of positive real numbers. Recall the interpretation of vertex and face labels as tuples with entries in $\mathbb{R}_+$. A vertex label is a V-tuple. Since K is a combinatorial torus, $F = 2V$. Since each face has three vertices, it follows that a face label is a $6V$-tuple. Choose some ordering of tuple entries for each of these types of labels and let

$$\mathcal{M} : \mathbb{R}_+^V \times \mathbb{R}_+^{6V} \to \mathbb{R}_+^{6V}$$

be the function determined by the the chosen ordering and the definition of multiplication of a vertex label by a face label in Section 3.3.

Lemma 5.3.2. *The (face label) angle error function*

$$E : \mathbb{R}_+^{6V} \to [0, \infty)$$

is continuous.

Proof. This follows from the definition of E and the continuity of angle sums. $\square$

Lemma 5.3.3. *The function*

$$E \circ \mathcal{M} : \mathbb{R}_+^V \times \mathbb{R}_+^{6V} \to [0, \infty)$$

is continuous.

Proof. It is clear from the definition that $\mathcal{M}$ is continuous, so the result follows from Lemma 5.3.2. $\square$

Notation. If R is a vertex label, let

$$\max R$$

be the maximum over $v \in K$ of $R(v)$. If S is a face label, $\max S$ will be similarly defined.

Notation. If R and R' are vertex labels, let

$$\max |R - R'|$$

be the maximum over $v \in K$ of $|R(v) - R'(v)|$. If S and S' are face labels, $\max |S - S'|$ will be similarly defined.

Lemma 5.3.4. *For each $\epsilon > 0$ there is a $\delta > 0$ such that for all $A', B' \in \mathbb{R}_+$,*

$$|A - A'|, |B - B'| < \delta \Rightarrow \max |S - S'| < \epsilon.$$

Proof. By construction the only entries appearing in the face label for A and B are 1, A, B, and AB. The result easily follows. $\square$

Proposition 5.3.5. *For each $\epsilon > 0$ there is a $\delta > 0$ such that for all $A', B' \in \mathbb{R}_+$,*

$$|A - A'|, |B - B'| < \delta \Rightarrow E(R' \cdot S) < \epsilon.$$

Proof. Fix $\epsilon > 0$. Since the entries of S are all positive, there is an open set $\mathcal{U} \subset \mathbb{R}_+^{6V}$ such that $S \in \mathcal{U}$, $\overline{\mathcal{U}} \subset \mathbb{R}_+^{6V}$, and $\overline{\mathcal{U}}$ is compact. Since $\mathcal{U}$ is open, by Lemma 5.3.4 there is a $\delta_1 > 0$ such that:

$$|A - A'|, |B - B'| < \delta_1 \Rightarrow S' \in \overline{\mathcal{U}}. \tag{5.35}$$

It follows from Lemma 5.3.1 that there is a $\delta_2 > 0$ such that:

$$|A - A'|, |B - B'| < \delta_2 \Rightarrow R'(v) \in [l, L] \text{ for all } v \in K. \tag{5.36}$$

Let

$$\mathcal{V} = [l, L]^V \times \overline{\mathcal{U}}.$$

By Lemma 5.3.1, $l > 0$. So

$$\mathcal{V} \subset \mathbb{R}_+^V \times \mathbb{R}_+^{6V}.$$

Observe that $\mathcal{V}$ is compact. It follows by Lemma 5.3.3 that $E \circ \mathcal{M}$ is uniformly continuous on $\mathcal{V}$. So there is a $\gamma > 0$ such that:

$$x, y \in \mathcal{V} \text{ and } |x - y| < \gamma \Rightarrow |E \circ \mathcal{M}(x) - E \circ \mathcal{M}(y)| < \epsilon. \tag{5.37}$$

By Lemma 5.3.4 we may choose δ so that:

$$0 < \delta < \min\{\delta_1, \delta_2\},$$

and

$$|A - A'|, |B - B'| < \delta \Rightarrow \max |S - S'| < \frac{\gamma}{L\sqrt{7V}}.$$

Now suppose that $|A - A'|, |B - B'| < \delta$. By choice of $\mathcal{U}$, choice of δ, and (5.35), both S and S' are in $\overline{\mathcal{U}}$. Moreover, by choice of δ and (5.36), $R' \in [l, L]^V$. It follows that:

$$|A - A'|, |B - B'| < \delta \Rightarrow (R', S), (R', S') \in \mathcal{V}. \tag{5.38}$$

By choice of δ and (5.35),

$$\begin{aligned} |A - A'|, |B - B'| < \delta &\Rightarrow \max R' \cdot \max |S - S'| < L \cdot \frac{\gamma}{L\sqrt{7V}}, \\ &\Rightarrow \max |R' \cdot S - R' \cdot S'| < \frac{\gamma}{\sqrt{7V}}, \\ &\Rightarrow |(R', S) - (R', S')| < \gamma. \end{aligned} \tag{5.39}$$

By (5.37), (5.38), and (5.39),

$$\begin{aligned} |A - A'|, |B - B'| < \delta &\Rightarrow |E \circ \mathcal{M}(R', S) - E \circ \mathcal{M}(R', S')| < \epsilon, \\ &\Rightarrow |E(R' \cdot S) - E(R' \cdot S')| < \epsilon, \\ &\Rightarrow E(R' \cdot S) < \epsilon. \end{aligned}$$

Note that the final implication follows since $R' \cdot S'$ is an affine packing label. $\square$

Remark. In Section 5.1, the quantities $M(R)$ and $\|R - \widetilde{R}\|$ were defined for a vertex label R. These quantities will be defined for face labels once a face label playing the role of $\widetilde{R}$ is specified. As in Section 3.3, let $S = S(\Gamma, A, B)$, and let R be a vertex label such that $R \cdot S$ is an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$. The required face label will be $R \cdot S$, scaled to agree with a given face label at a fixed vertex v_0 in a fixed face f_0.

Proposition 5.3.6. *For each $\epsilon > 0$ there is a $\delta > 0$ such that:*

$$E(R' \cdot S) < \delta \Rightarrow \|R'S - RS\| < \epsilon.$$

for any vertex label R'.

Proof. As in Section 3.1 and Section 3.4, we reinterpret the arguments from Section 5.2 in such a way that they are valid for face labels. In particular, we apply them to the affine packing label $R \cdot S$ to obtain this statement (note that Theorem 5.2.19 is a statement about the vertex packing label $\widetilde{R}$). $\square$

Notation. Let $\mathcal{F} : \mathbb{R}_+^2 \times \mathbb{R}_+^V \to \mathbb{R}^{\widehat{V}}$ be the function that takes side-pairing moduli A and B and vertex label R to the vertex label $\widehat{R}$ as in Section 4.1.

Notation. Fix a layout normalization for $\widehat{K}$ (for example, a chosen circle has center at the origin and a chosen neighboring circle has center on the positive real axis) for two neighboring vertices and a layout order such that each vertex after the first two is a neighbor to two previous vertices and let $\mathcal{Z} : \mathbb{R}_+^{\widehat{V}} \to \mathbb{C}^{\widehat{V}}$ be the center function that maps a vertex label for $\widehat{K}$ to the centers of circles for the circle arrangement (not necessarily a circle packing) determined by the layout order.

Notation. Let $\mathcal{A} : \mathbb{C}^{\widehat{V}} \to \mathbb{P}$ be the function taking the "centers" z_i and z_i' $(i = 1, 2)$ to the side-pairing parameter $\alpha = (z_2' - z_1')/(z_2 - z_1)$. Similarly, we may define the function $\mathcal{B} : \mathbb{C}^{\widehat{V}} \to \mathbb{P}$ taking "centers" to the side-pairing parameter β.

Notation. Let $\mathcal{S} : \mathbb{C}^{\widehat{V}} \to \mathbb{P}^2$ be the function taking "centers" to side-pairing parameters α and β. So $\mathcal{S} = \mathcal{A} \times \mathcal{B}$.

Lemma 5.3.7. *The functions $\mathcal{F}$, $\mathcal{Z}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$ are continuous.*

Proof. Clear from the definitions of the functions. $\square$

Notation. Let $\widetilde{\mathcal{F}} : \mathbb{R}_+^2 \to \mathbb{R}_+^V$ be the function mapping a pair of affine factors A and B to the vertex label R such that $R \cdot S$ is an affine packing label for K that is $\Gamma_1(A)$ and $\Gamma_2(B)$.

The main result of this section shows that the side-pairing parameters are continuous in the side-pairing moduli (which are exactly the affine factors).

Theorem 5.3.8. *For each $\epsilon > 0$ there is a $\delta > 0$ such that:*

$$|A - A'|, |B - B'| < \delta \Rightarrow |\alpha - \alpha'|, |\beta - \beta'| < \epsilon.$$

Proof. By Lemma 5.3.7, the function $\mathcal{S} \circ \mathcal{Z} \circ \mathcal{F}$ is continuous. Hence it suffices to show that $\widetilde{\mathcal{F}}$ is continuous. Let $\epsilon > 0$. By Lemma 5.3.1 there exist positive numbers δ_1 and L such that:

$$|A - A'|, |B - B'| < \delta_1 \Rightarrow R(v), R'(v) \leq L \tag{5.40}$$

for all $v \in K$. By Proposition 5.3.6, we may choose a $\gamma > 0$ such that:

$$E(R' \cdot S) < \gamma \Rightarrow \|R'S - RS\| < \frac{\epsilon}{L \cdot \max S} \tag{5.41}$$

for any vertex label R'. By Proposition 5.3.5, we may choose a $\delta_2 > 0$ such that:

$$|A - A'|, |B - B'| < \delta_2 \Rightarrow E(R' \cdot S) < \gamma. \tag{5.42}$$

Now let $\delta = \min\{\delta_1, \delta_2\}$. Then by choice of δ, (5.40), (5.41), and (5.42),

$$\begin{aligned}
|A - A'|, |B - B'| < \delta &\Rightarrow E(R' \cdot S) < \gamma, \\
&\Rightarrow \|SR' - SR\| < \frac{\epsilon}{L \cdot \max S}, \\
&\Rightarrow \|R' - R\| < \frac{\epsilon}{L}, \\
&\Rightarrow \max |R' - R| < \epsilon.
\end{aligned}$$

It follows that $\widetilde{\mathcal{F}}$ is continuous. $\square$

Chapter 6

Future Directions

6.1 Experimental Observations

For a combinatorial torus K, fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$ for K, and positive affine factors A and B, there is an associated circle packing on an affine torus. Using the results of Section 4.4 it is possible to identify the affine parameter c and Teichmüller parameter ω of this affine torus. Let

$$\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C} \times \mathbb{H}$$

be the map that takes (A, B) to the pair of affine and Teichmüller parameters of the associated affine torus. Let

$$\pi : \mathbb{C} \times \mathbb{H} \to \mathbb{H}$$

be projection onto the second coordinate. The following questions, motivated by results in Mizushima (2000), are open.

Question. *For a fixed combinatorial torus K and fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$ for K, is the map $\pi \circ \phi$ surjective?*

Question. *For a fixed combinatorial torus K and fundamental pair $\Gamma = \Gamma_1 * \Gamma_2$ for K, is the map $\pi \circ \phi$ 2-to-1 except at a single point?*

Any control over the affine and Teichmüller parameters in terms of the affine factors A and B might prove useful in future research on these questions. The continuity result in Section 5.3 provides a type of the desired control. Experimental observations of certain monotonicities may be useful for establishing stronger control. Note that these observations are made with respect to packings in normalized position (and the notational conventions of Section 4.1 are employed).

Experimental Observation 1. If $A > 1.0$ and $B = 1.0$, and if B is kept fixed while A increases, then:

1. The radius of the circle centered at $z_1 = 1$ increases,
2. The circle center z_n travels further around the unit circle in the negative direction.

Figures 6.1 through 6.9 show examples of such experimantal observations.

Experimental Observation 2. If $A < 1.0$ and $B = 1.0$, and if B is kept fixed while A decreases, then:

1. The radius of the circle centered at $z_1 = 1$ increases,
2. The circle center z_n travels further around the unit circle in the positive direction.

Figures 6.10 through 6.17 show examples of such experimental observations. Analogous observations hold for when $A = 1.0$ and B varies.

Experimental Observation 3. If $A = 1.0$ and $B > 1.0$, and if A is kept fixed while B increases, then:

1. The radius of the circle centered at $z_1 = 1$ increases,
2. The circle center y_1 travels further around the unit circle in the positive direction.

Experimental Observation 4. If $A = 1.0$ and $B < 1.0$, and if A is kept fixed while B decreases, then:

1. The radius of the circle centered at $z_1 = 1$ increases,
2. The circle center y_1 travels further around the unit circle in the negative direction.

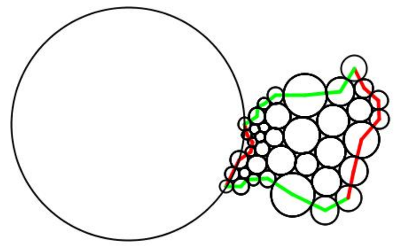

Figure 6.1: $A = 2.0,\ B = 1.0$

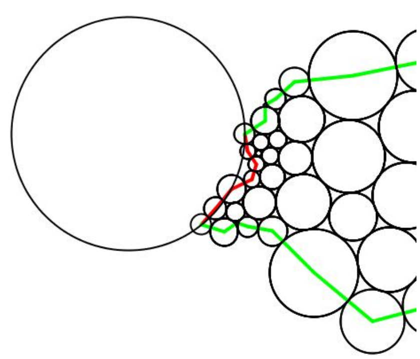

Figure 6.2: $A = 3.0,\ B = 1.0$

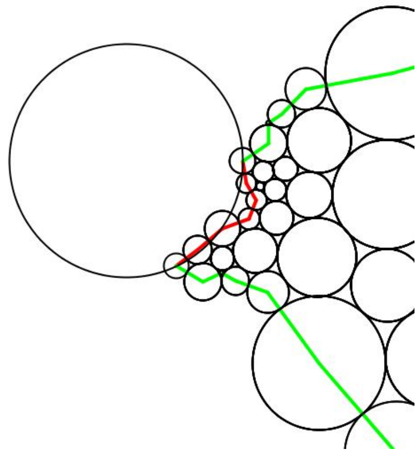

Figure 6.3: $A = 4.0$, $B = 1.0$

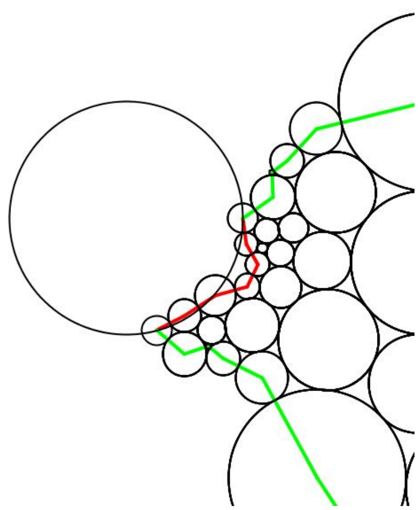

Figure 6.4: $A = 5.0$, $B = 1.0$

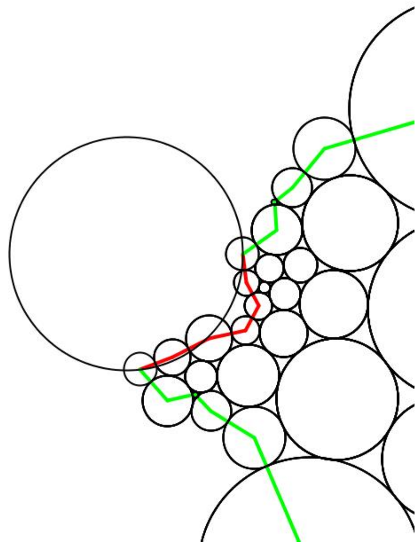

Figure 6.5: $A = 6.0$, $B = 1.0$

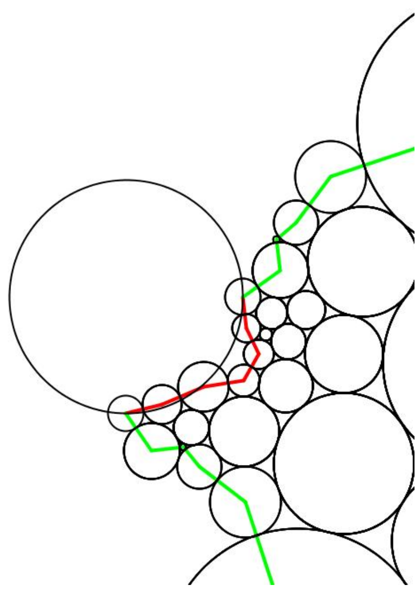

Figure 6.6: $A = 7.0$, $B = 1.0$

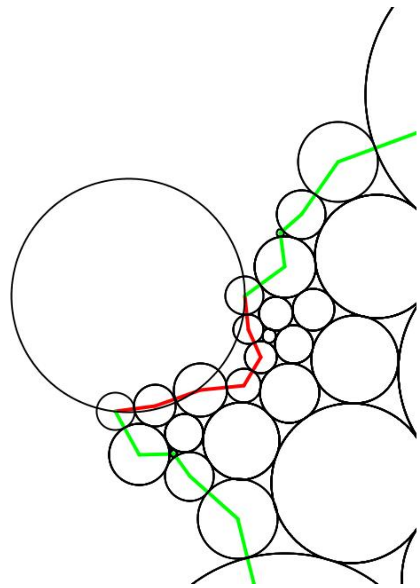

Figure 6.7: $A = 8.0$, $B = 1.0$

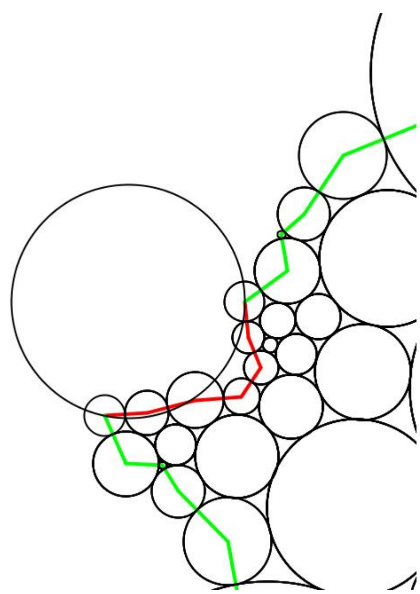

Figure 6.8: $A = 9.0$, $B = 1.0$

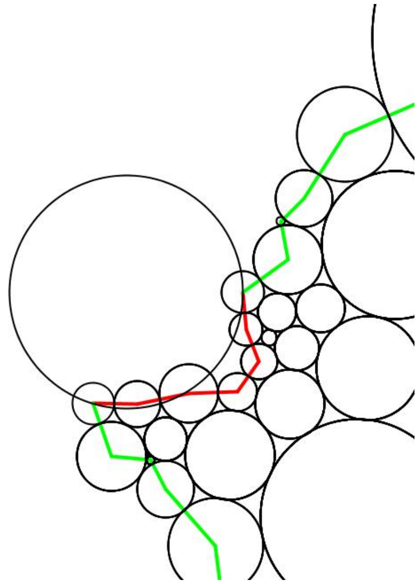

Figure 6.9: $A = 10.0$, $B = 1.0$

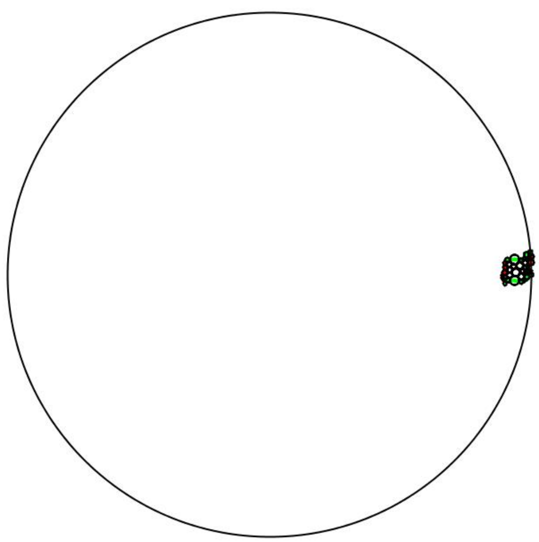

Figure 6.10: $A = 0.9$, $B = 1.0$

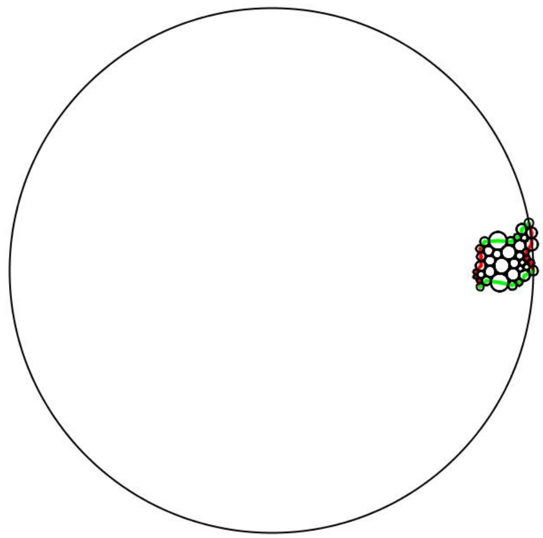

Figure 6.11: $A = 0.8$, $B = 1.0$

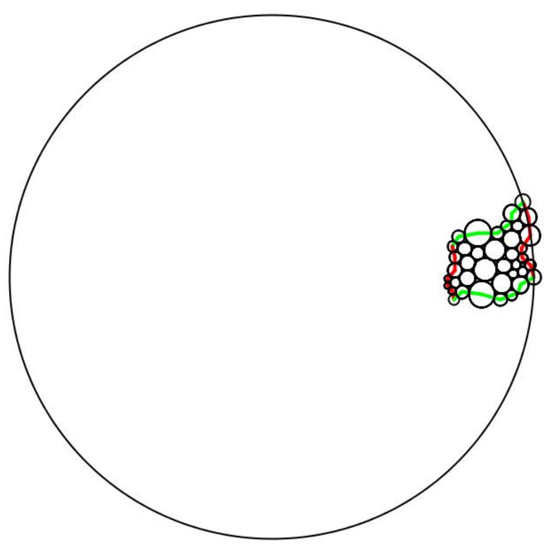

Figure 6.12: $A = 0.7$, $B = 1.0$

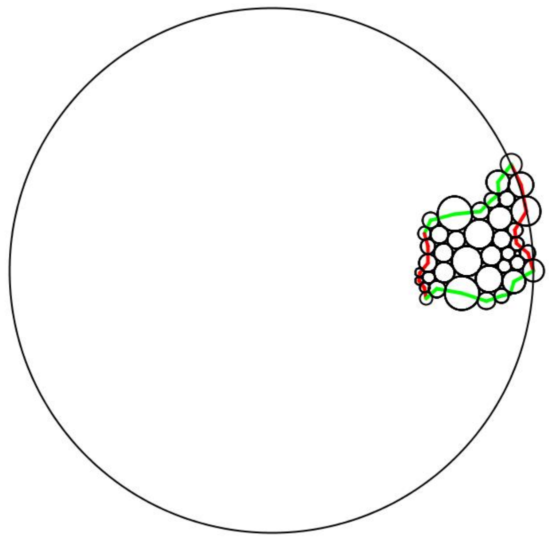

Figure 6.13: $A = 0.6,\ B = 1.0$

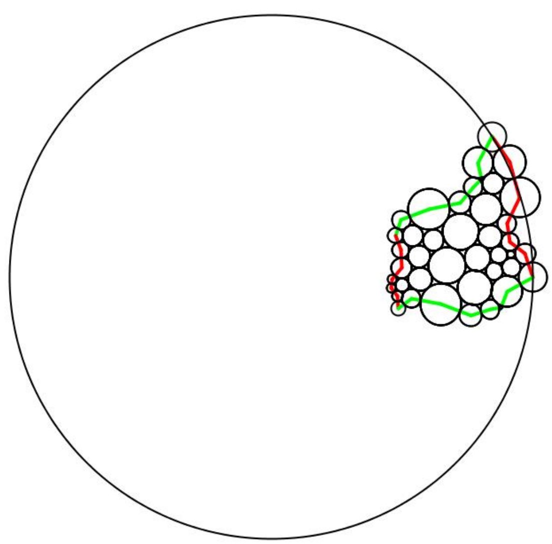

Figure 6.14: $A = 0.5,\ B = 1.0$

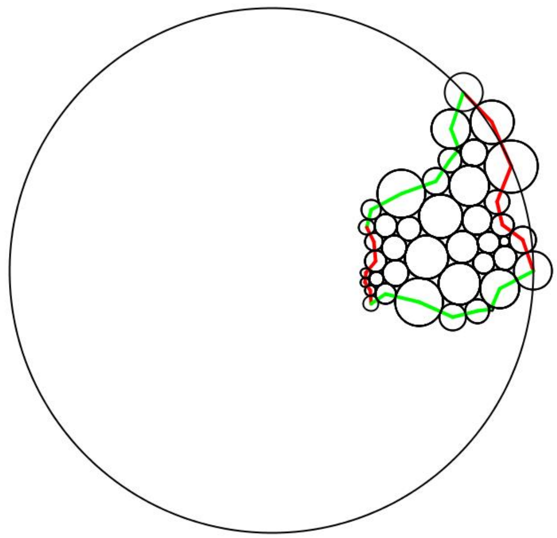

Figure 6.15: $A = 0.4$, $B = 1.0$

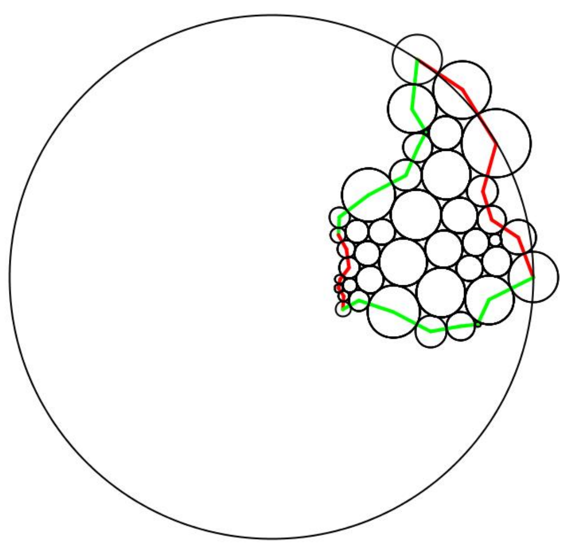

Figure 6.16: $A = 0.3$, $B = 1.0$

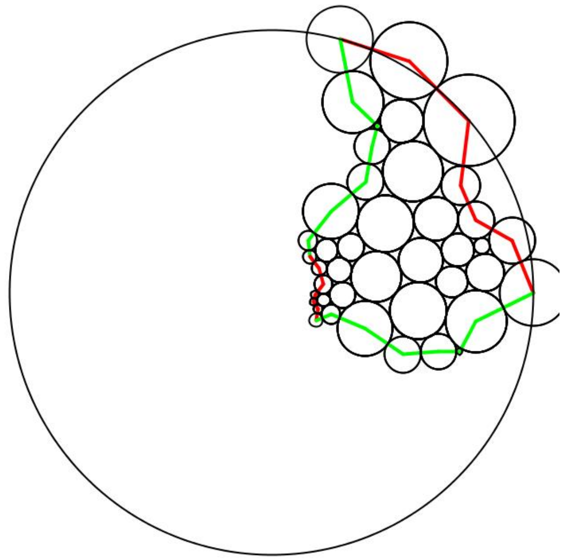

Figure 6.17: $A = 0.2$, $B = 1.0$

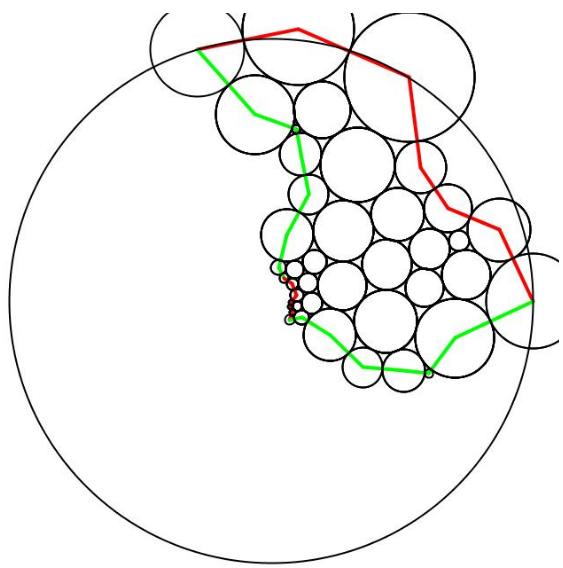

Figure 6.18: $A = 0.1$, $B = 1.0$

6.2 Branched Affine Packing Labels for Higher Genus Surfaces

A **branched** packing label has angle sums $2\pi(\beta_v + 1)$ at each vertex $v \in K$, where $\beta_v \geq 0$ is an integer. This definition applies to vertex and face labels. The angle relative to a label at v in any face f containing v must be less than π, so a necessary condition for branching is that v have at least $2\beta_v + 3$ neighbors.

This necessary condition is not in general a sufficient condition, however, since angles at more than a single flower must taken into account. A necessary and sufficient condition for the existence of branched vertex packing labels is given in Stephenson (2005) for the case when K is a combinatorial closed disc. It involves the notion of a branch structure.

Definition 6.2.1. Let K be a combinatorial closed disc, and let $\beta = \{b_1, \ldots, b_m\}$ be a list of interior vertices of K, including possible repetitions. If the following condition holds, then β is a **branch structure** for K: for each simple, closed, positively oriented edge path γ in K consisting of k edges, $k > 2N + 2$, where N is the number of points of β inside γ, counting repetitions.

The theorem in Stephenson (2005) states that if β is formed by listing each vertex $v \in K$ exactly β_v times, then there is a branched packing label for K having angle sums $2\pi(\beta_v + 1)$ if and only if β is a branch structure for K.

Now suppose that K is a combinatorial closed surface of genus $g > 1$. Since K is closed, $3F = 2E$, so $V - E + F = V - F/2$. The Euler characteristic for K is $\chi(K) = 2 - 2g$, and hence:

$$\begin{aligned} F/2 &= V + 2g - 2, \\ \pi F &= 2\pi V + 2\pi(2g - 2). \end{aligned} \tag{6.1}$$

Observe that $2g - 2 > 0$ since $g > 1$. Any euclidean label (whether vertex or face) for K must have total angle πF, so it follows from $2g - 2 > 0$ and (6.1) that there is no euclidean packing label for K (that is, no label with angle sums 2π at every vertex). But since $2g - 2 > 0$, (6.1) allows the possibility that there is a branched packing label for K. Three questions naturally arise.

Question. *If K is a combinatorial closed surface of genus $g > 1$, what are the necessary and sufficient combinatorial conditions for the existence of a branched euclidean vertex packing label for K?*

Question. *Can these conditions be reinterpreted in such a manner that they apply to branched face packing labels for* K*?*

Question. *If branched face packing labels for* K *exist, what is their geometric significance?*

Bibliography

Andreev, E. M. (1970a). Convex polyhedra in Lobacevskii space. *Math. USSR Sbornik*, 10:413–440.

Andreev, E. M. (1970b). Convex polyhedra of finite volume in Lobacevskii space. *Math. USSR Sbornik*, 12:255–259.

Beardon, A. F. and Stephenson, K. (1990). The uniformization theorem for circle packings. *Indiana Univ. Math. J.*, 39(4).

Gunning, R. C. (1981). Affine and projective structures on Riemann surfaces. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 225–244. Princeton Univ. Press, Princeton, N.J.

He, Z.-X. and Schramm, O. (1993). Fixed points, Koebe uniformization and circle packings. *Annals of Math.*, 137:369–406.

Koebe (1936). Kontaktprobleme der Konformen Abbildung. *Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl.*, 88:141–164.

Kojima, S., Mizushima, S., and Tan, S. P. (2003). Circle packings on surfaces with projective structures. *J. Differential Geom.*, 63(3):349–397.

Kojima, S., Mizushima, S., and Tan, S. P. (2006). Circle packings on surfaces with projective structures and uniformization. *Pacific J. Math.*, 225(2):287–300.

Milnor, J. (1958). On the existence of a connection with curvature zero. *Comment. Math. Helv.*, 32:215–223.

Mizushima, S. (2000). Circle packings on complex affine tori. *Osaka J. Math.*, 37(4):873–881.

Rodin, B. (1991). On a problem of A. Beardon and K. Stephenson. *Indiana Univ. Math. J.*, 40:271–275.

Stephenson, K. (1992-2011). `CirclePack` software. `http://www.math.utk.edu/~kens`.

Stephenson, K. (2005). *Introduction to circle packing*. Cambridge University Press, Cambridge.

Thurston, W. (1985). The finite Riemann mapping theorem. Invited talk, An International Symposium at Purdue University in celebrations of de Branges' proof of the Bieberbach conjecture, March 1985.

Vita

Christopher Thomas Sass graduated *Summa Cum Laude* from Middle Tennessee State University in 1998 with a Bachelor of Arts Degree and a double major in Mathematics and Philosophy. He earned a Master of Arts degree in Philosophy from the University of Notre Dame in 2001. After six years in the graduate program in Mathematics at the University of Tennessee, he will graduate in August 2011 with a Doctor of Philosophy degree. In August 2011 he will join the faculty of Young Harris College in Georgia as an Assistant Professor in Mathematics. He is married to Bethann Bowman.